AN INTENSIONAL LEIBNIZ SEMANTICS FOR ARISTOTELIAN LOGIC

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Abstract. Since Frege's predicate logical transcription of Aristotelian categorical logic, the standard semantics of Aristotelian logic considers *terms* as standing for *sets of individuals*. From a philosophical standpoint, this *extensional* model poses problems: There exist serious doubts that Aristotle's *terms* were meant to refer always to *sets*, that is, entities composed of individuals. Classical philosophy up to Leibniz and Kant had a different view on this question—they looked at *terms* as standing for *concepts* ("Begriffe"). In 1972, Corcoran presented a formal system for Aristotelian logic containing a calculus of natural deduction, while, with respect to semantics, he still made use of an extensional interpretation. In this paper we deal with a simple *intensional* semantics for Corcoran's syntax—*intensional* in the sense that no individuals are needed for the construction of a *complete* Tarski model of Aristotelian syntax. Instead, we view concepts as containing or excluding other, "higher" concepts—corresponding to the idea which Leibniz used in the construction of his characteristic numbers. Thus, this paper is an addendum to Corcoran's work, furnishing his formal syntax with an adequate semantics which is free from presuppositions which have entered into modern interpretations of Aristotle's theory via predicate logic.

§1. Introduction.

1.1. Historical remarks. In 1972 Corcoran presented a formal system for Aristotelian logic containing a calculus of natural deduction (Corcoran, 1972b). The basic building blocks of this system are *terms*, as in Aristotle's works, as well as in those of his successors up to but not including Frege.[1] Corcoran's whole system does not contain variables or constants for *individuals*, and this conforms to the mainstream of the tradition in Aristotelian logic from Aristotle himself via Leibniz and Kant up to the first "modern" term logical system of Łukasiewicz (1957).

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[1] In his "*Begriffsschrift*" Frege (1967) laid the foundations of modern predicate logic, where he introduced a by now classical transcription of Aristotelian logic by means of quantification on individuals: The universal affirmative proposition "*All S is P*" is, in our modern notation, translated into $\forall x : Sx \rightarrow Px$, and "*Some S is P*" into $\exists x : Sx \wedge Px$. A major objection to this modeling of Aristotelian syntax is that it does not exactly reproduce the Aristotelian theorems; more specifically: By the rules of First-Order Predicate Logic one cannot, for example, prove the *Law of Subalternation*, $A(S, P) \longmapsto I(S, P)$, which plays a central role in Aristotle's theory. Whereas our undergraduate texts today still use to offer a simple "solution" to this problem, called *existential import*, we know now that such auxiliary constructs have nothing to do with problems of Aristotelian logic, but solely of its inadequate translation into a modern framework. I agree with Nedzynski (1979): "The problem of existential import developed along with the development of modern symbolic logic during the nineteenth century. The problem is peculiar to the standard predicate calculus. There never was a real problem of existential import within the traditional syllogistic logic—it was placed there in retrospect by the modern logicians."

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In Łukasiewicz's as well as in Corcoran's *term logic*, the proposition "*All x is y*" is denoted by *Axy*, where *x* and *y* stand for Aristotelian *term*s like *man*, *animal*, *soul*, and so forth.[2] In contrast to Frege's First-Order Logic (FOL) formalization, *term logic* allows two different types of interpretation, namely, *intensional* and *extensional* ones. This falls into line with a vast literature on the question whether Aristotle's *terms* were meant to stand for *concepts* or for *sets of individuals (objects)*. We will not go into the historical details of these discussions (see Frisch, 1969) but just mention here that, since medieval times, the use of a term has been split into its *extensional* and *intensional* aspects (see the following Section 1.2).

Corcoran and Martin (1997), in their works on the completeness theorem for Aristotelian logic, have chosen differing *extensional* interpretations. The justification for this naming arises from the fact that, in these interpretations, each term is mapped to a certain set $e(x)$, and "*All x is y*" (Axy) is defined to be true *iff*

$$e(x) \subseteq e(y). \tag{1.1}$$

This says that, in case of Axy being true, y has at least as great an extension as x: The more general a concept is, the bigger is its extension.

From a historical and philosophical standpoint (see the following Section 1.2), it is interesting to ask how an *intensional interpretation* of the ancient term logic could be constructed. In such an intensional model, terms would be mapped onto sets $i(x)$ standing for sets of intensions or meanings *contained* in x, and the truth of "*All x is y*," Axy, would now be equivalent to

$$i(x) \supseteq i(y). \tag{1.2}$$

Thus this type of interpretation would reflect the idea that the intensional content *grows* while proceeding to more and more special concepts, which is contrary to the way extensions behave.

Leibniz was fully aware of the difference of these two methods of interpretation of Aristotle's logic, and he was the first to construct an intensional interpretation, if only in terms of his characteristic numbers. It was his idea to use pairs of numbers which has lead us to the general construction of a set theoretical intensional model of Aristotelian logic. Leibniz of course did not put his ideas into the modern framework of syntax/semantics interplay of the Tarski type, as we will do, but in hindsight it has been a big step in that direction (Glashoff, 2002).

In the following section, we will first refer to some historical and technical details of the concepts *extension* and *intension*, and we will also point out the intrinsic difficulty of constructing intensional models of Aristotelian logic.

1.2. Extension and intension. The dichotomy of *extension* and *intension* (or, of *extension* and *meaning* (Quine, 1951), or, of extension and *comprehension* (Frisch, 1969), or, of *reference* and *meaning* (Frege, 1967); in linguistics, the correponding naming is *denotation* and *connotation* (Lyons, 1977)) has been a recurring major theme in ancient European logic since Aristotle.[3] This theme goes back to Aristotle's Theory of Categories, which makes use of the terms *genus* and *species* in order to describe the relation between

[2] In fact, Łukasiewicz's makes use of his inconvenient infix notation, while Corcoran's notation deviates only slightly from the one used here.

[3] "The Aristotelian notion of essence was the forerunner, no doubt, of the modern notion of intension or meaning." (Quine, 1951, p. 22).

higher (*genus*) and lower (*species*) terms, and it appeared later in the shape of Porphyrian trees.[4]

There is a long history of the development of a precise concept of extension and intension of a term (or a concept) from Porphyry (Emilsson, 2009; Porphyry, 1975) to Arnauld and Nicole in their *Port-Royal Logic*.[5] Leibniz and, later, Kant, defined the *extension (Umfang)* of a concept to consist of those concepts that *fall under* the given concept; the *intension (Inhalt)* of a concept are those concepts that occur *within* the given concept. For example, to the *extension* of the concept "deciduous tree" there belong all the different types of deciduous trees like "birch," "cottonwood," and so forth, and to the *intension* of "deciduous tree" belong the—more general—concepts "tree," "plant," "casting leaves in winter," and so forth. Kant also formulated his "Law of reciprocity":

> "Intension and extension of a concept stand to each other in an reciprocal relation. That is to say, the more a concept contains *under* it, the less it contains *in* itself, and vice versa."[6]

Leibniz' and Kant's aim of a precise definition of *extension* and *intension* was to provide a sound basis for an interpretation of syllogistic logic.[7]

While these two possibilities of defining a semantics for the Aristotelian universal positive proposition Axy (see (1.1), (1.2)) are quite symmetrical and, by this symmetry, do not seem to present any conceptual difficulty, there is an intrinsic complication caused by the other types of Aristotelian propositions.

In an extensional interpretation, Exy ("*No x is y*") is true if and only if the extensions of x and y do not overlap; that is, if and only if there is no concept z which falls under x as well as under y:

$$e(x) \cap e(y) = \emptyset. \tag{1.3}$$

By negation, Ixy ("Some x is y") holds true if and only if there is some concept z which falls under x as well as under y.[8] For example, the truth of "Some boats are houses" depends on whether there is, within the semantic domain chosen, a concept "houseboat" which falls under "house" as well as under "boat."

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[4] The Theory of Categories is a "rich" theory, with many different types (categories) of predicates, and with different types of relations between elements of these different categories: essential and accidental predication, *genus proxima* and *differentia specifica* or, as Grice named it, Izzing and Hazzing (Code, 1986). In contrast to that, syllogistics—since the *Analytica Priora* up to Kant—is "flat": only one type of term—just terms, no further qualification.

[5] The terms corresponding to extension and intension in Arnauld (1861) are *étendue* and *compréhension.*

[6] Kant (1991), my translation of: "Inhalt und Umfang eines Begriffs stehen gegeneinander in umgekehrten Verhältnis. Je mehr nämlich ein Begriff *unter* sich enthält, desto weniger enthält er *in* sich und umgekehrt." An older translation by Richardson (Kant, 1836) reads: "The matter and the sphere of a conception bear one another a converse relation. The more a conception contains under it, the less it contains in itself; and *vice versa*."

[7] While "Kant bashing" is a common exercise in circles of modern logicians, we agree with Tolley's assessment, "…that many widely-held beliefs about Kant's views on logic are gravely mistaken and unfounded. I have in mind here primarily the beliefs that: (1) Kant simply inherits and repeats what the tradition has taught about logic since Aristotle, (2) his logical doctrines carry little weight in his philosophical system, and (3) his views have been so thoroughly superceded by more recent work (e.g., by Frege) that they are unable to contribute anything to contemporary debates." (Tolley, 2007).

[8] Aristotle sometimes argued in this way, called *ecthesis*, in his *Analytica Priora*.

If we try to define a similar semantics with respect to intensional logic, we run into problems. How should we interpret Ixy solely in terms of the intensions of x and y? The overlapping of *intensions*, that is, of concepts *higher than* x and y, taken as criterion for the truth of Ixy, will certainly not work, which is shown by the following example: The concepts "red" and "green" have the common "higher" concept "color," but of course Ixy will not be true in any reasonable domain. On the other hand, it is not difficult to imagine situations where two concepts share no common superconcept, but nevertheless have a common subconcept.

Thus, apparently there is an "extensional bias" built into the syntax of Aristotelian logic. This is due to Aristotle's choice of "downward quantors" E and I which impedes a simple intensional definition of the truth of Exy and Ixy.

It was Leibniz who, after trying for many years, found an ingenious solution to this difficulty. Put into the formalism of his characteristic numbers, he associated, with each concept x, *two sets* of other concepts: The first one consists, as usual, of all concepts higher than x (it corresponds to all y such that Axy holds). The second one consists of all concepts which are definitely *not* contained in x, corresponding to all y such that Exy holds.

We will use this device in our definition of an intensional interpretation, and, making use of this construction of pairs of sets of intensions, it will be possible to prove a completeness theorem for intensional models of Aristotle's term logic. As a preparation for the completeness theorem, we will now present an introduction to the formal concept of interpretation and models of Aristotelian term logic.

> **Let:** T denote the set of *categorical terms,* $T = \{x, y, z, \ldots\}$, and let A, I, E, O denote four logical constants. An Aristotelian *sentence* or *proposition* is any expression of the form $U\xi\upsilon$, where U is one of the logical constants, and ξ, υ are categorical terms. Let $\mathcal{L}$ denote the set of all sentences or propositions, and let P denote a fixed subset of $\mathcal{L}$.

1.2.1. Extensional semantics By an *extensional interpretation e* of Aristotelian term logic we will denote the following construction. Let $\mathcal{O}$ be a set of nonempty subsets of a given basic set M.[9] e is a function from T into $\mathcal{O}$; that is, e assigns a nonempty subset $X = e(x)$ of M, belonging to $\mathcal{O}$, to each term $x \epsilon T$. This function will be extended to a function on $\mathcal{L}$ by defining

$$\begin{aligned} e(Axy) &= \mathit{true} \quad \mathit{iff} \quad e(x) \subseteq e(y) \\ e(Oxy) &= \mathit{true} \quad \mathit{iff} \quad e(Axy) = \mathit{false} \\ e(Exy) &= \mathit{true} \quad \mathit{iff} \quad e(x) \cap e(y) = \emptyset \\ e(Ixy) &= \mathit{true} \quad \mathit{iff} \quad e(Exy) = \mathit{false}. \end{aligned}$$

For a set $P \subset \mathcal{L}$ of sentences, e is said to be a *true interpretation*, if $e(d) = \mathit{true}$ for all $d \epsilon P$.

There are different methods of constructing extensional interpretations. Here we will present the construction of the Henkin-style interpretation of Martin (1997)[10] which is

[9] Most often, $\mathcal{O}$ will be the power set of M, that is, $\mathcal{O} = 2^M$.

[10] Martin's work is an extension and a generalization of Corcoran's. The construction of his extensional interpretation makes use of the concept of *saturation*, well known from the

easier to compare with our intensional interpretation to be presented later, than Corcoran's original one.

This method of constructing an extensional true interpretation for a given set P of propositions goes roughly as follows. We choose $\mathcal{O} = 2^T$, that is, the set of all subsets of the set of terms T and define, for $x \in T$,

$$e(x) = \{x\} \cup \{y \in T \mid Ayx \in P\}. \tag{1.4}$$

This interpretation function assigns to each term x in T the set of those terms which are "under" x with respect to A; thus it makes sense to name $e(x)$ the *extension* of x.[11] It is easy to see that e as defined in (1.4) satisfies the relation (1.1).

1.2.2. Intensional interpretation: Leibniz's characteristic numbers The most obvious idea for defining an *intensional* interpretation would, by symmetry with (1.4), be

$$i^\star(x) = \{x\} \cup \{y \in T \mid Axy \in P\}. \tag{1.5}$$

This, however, does not work because—as we tried to explain above—it is not possible to properly define the truth of $i^\star(Exy)$ and $i^\star(Ixy)$ by referring to "superconcepts" $i^\star(x)$ and $i^\star(y)$ alone. Within his framework of characteristic numbers, Leibniz overcame the problem as follows. He considered a situation where, within the set T of terms, there exists a denumerable set $\hat{T} \subset T$ of *elementary* or *simple* terms which he numbered using the sequence of prime numbers from 2 on:

$$\begin{array}{rccccc} \hat{T}: & x_1 & x_2 & x_3 & x_4 & \dots \\ & 2 & 3 & 5 & 7 & \dots \end{array}$$

Now each finite subset of $\hat{T}$ can be characterized by a unique number; for example, to the subset $\{x_2, x_4\}$ there will be assigned the number $3 \times 7 = 21$.[12] Because of the unique prime factorization theorem, also called Fundamental Theorem of Arithmetic, there exists, for each number n, a unique finite subset of $\hat{T}$, composed of all those simple terms x_i which belong to the prime factors of n.[13] Given a general, nonsimple term $x \in T$, the characteristic number of x, written by Leibniz as

$$+s - \sigma \tag{1.6}$$

is defined as follows[14]: s is the product of prime numbers of elementary properties *contained* in x, that is, of all elementary properties y such that Axy holds true. On the other

completeness proof of FOL. In the Aristotelian context, saturation is directly connected to Aristotle's *ecthesis.* See also the alternative completeness theory by Smith (1983), based on *ecthesis.*

11 Let us remark that this definition will work properly only for *maximal consistent saturated* sets P (Martin (1997)), a concept which we will introduce in the next section.

12 Leibniz's own example of how two terms combine intensionally into a term standing below these two, is as follows: "If, for example, we assume that the item 'animal' is expressed by means of the number 2 (or, in general, by a), and the item 'rational' by means of the number 3 (or, in general, by r), then 'man' is expressed by 2 × 3, i.e. 6, as the result of the product of 2 by 3 (or, in general, by the number a × r)." (My translation of Leibniz, 1999).

13 Multiple prime factors have to be ignored; that is, the number $63 = 3^2 \times 7$ denotes the same subset $\{x_2, x_4\}$as the number $21 = 3 \times 7$.

14 It is quite clear that $+s - \sigma$ is an inventive notation for what we today call an ordered pair of numbers, (s, σ). The $+/-$ sign added a lot of confusion to later readings of Leibniz's papers. For an extensive discussion of the details, see Glashoff (2002).

hand, σ is the product of all primes belonging to those elementary properties z such that Exz holds. In the context of our example from above: If there is a concept x characterized by $\{Axx_1, Axx_3, Exx_4\}$, then the characteristic number of x would be $+10 - 7$. In his groundbreaking book, Łukasiewicz (1957) made use of Leibniz' characteristic numbers within the context of the proof of his completeness theorem. Later, in his thesis, Maldonado (1998) also used this device for an arithmetical semantics of Corcoran's syntax. The aim of the present paper is to free Leibniz' concept from the algebraic context and thus to establish his ideas as a general tool for constructing intensional models of Aristotelian term logic.

Following this idea, let us define the interpretation function i a little more sophisticatedly than according to (1.5) by an ordered pair of subsets of T:

$$i(x) = (\{x\} \cup \{y \in T \mid Axy \in P\}, \{y \in T \mid Exy \in P\}) \tag{1.7}$$

$$= (s(x), \sigma(x)). \tag{1.8}$$

This function can now be extended to sentences by the following definitions:

$$i(Axy) = true \quad iff \quad s(x) \supseteq s(y) \quad and \quad \sigma(x) \supseteq \sigma(y)$$

and

$$i(Exy) = true \quad iff \quad s(x) \cap \sigma(y) \neq \emptyset \quad or \quad s(y) \cap \sigma(x) \neq \emptyset.$$

The interpretation of Oxy and Ixy results from negating Exy and Axy, respectively.

The rationale behind these definitions is: For $x \in T$, $s(x)$ collects the terms which are "positively" (by means of Axy) contained in x, and $\sigma(x)$ contains the terms "negatively" contained in x, that is, by means of Exy. The definition given above for the interpretation of Exy signifies that there is an overlap between the positive and negative components of x and y, respectively.

§2. Syntax. In this section we will give a short account of Aristotelian syntax in its modern form, based on a system of natural deduction as given by Corcoran (1972a, 1972b, 1974), and independently, by Smiley (1973).[15] We will also take into account the generalized version of Martin (1997) although keeping to Corcoran's somewhat simpler formalism.

Let us, as above, denote by $T = \{x, y, \ldots\}$ the Aristotelian *terms*, and by A, E, I, O the *logical constants*. $\mathcal{L}$, the *language* of Aristotelian logic, consists of all *propositions* $U\xi\upsilon$, where U is one of the four logical constants, and ξ, υ are different[16] arbitrary terms. For a proposition d, Cd denotes the contradictory of d, the definition of which is given by

$$C(Axy) := Oxy, \quad C(Ixy) := Exy, \quad and \quad CCd := d.$$

For a given set P of propositions, called the premises, the Aristotelian system allows for the generation of new, additional sentences not contained in P governed by the following rules.

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[15] Let us also refer to the German work of Kurt Ebbinghaus (1964), which reconstructs an "Aristotelian calculus" using Paul Lorenzen's Operative Logic.

[16] We agree with Corcoran (1972b) who points to the fact that, in Aristotle's works, there is a constant avoidance of sentences such as Axx and Ixx. Therefore we also do not allow these trivial sentences, although they would not constitute any formal difficulty.

DEFINITION 2.1. *The primitive rules of the syntax*[17] *are:*

(1) Conversions:

$$(C1) \quad Exy \vdash Eyx \quad (E\text{-}conversion)$$

$$(C2) \quad Axy \vdash Iyx \quad (partial\ A\text{-}conversion)$$

$$(C3) \quad Ixy \vdash Iyx \quad (I\text{-}conversion)$$

(2) Perfect syllogisms:

$$(PS1) \quad Axy, Ayz \vdash Axz \quad (Barbara)$$

$$(PS2) \quad Exy, Azx \vdash Ezy \quad (Celarent)$$

$$(PS3) \quad Axy, Izx \vdash Izy \quad (Darii)$$

$$(PS4) \quad Exy, Izx \vdash Ozy \quad (Ferio).$$

A *proof* or *deduction* of a proposition d from a set of premises P is of two different types:

DEFINITION 2.2.

(1) A direct deduction of a sentence d from P is a finite list of sentences, beginning with all or some sentences of P and ending with d, where each subsequent list element is

- *a repetition of a previous line*
- *a conversion of type* $(C1)$, $(C2)$, $(C3)$ *of a previous line*
- *a result of applying one of the perfect syllogisms* $(PS1)$, $(PS2)$, $(PS3)$, $(PS4)$ *to two previous lines.*

(2) An indirect deduction of a sentence d from P is defined as a direct deduction of a pair of contradictions e and $C(e)$ *from* $P \cup \{C(d)\}$.

A set of propositions is *inconsistent* if there are two deductions having all premises in the set and having contradictory conclusions. Otherwise, a set is *consistent*. A consistent set having no consistent supersets is *maximally consistent*.

LEMMA 2.3. (Corcoran, 1972b; Martin, 1997) *Let S be maximally consistent. Then the following hold:*

(1) $d \in S$ iff $S \vdash d$;
(2) $d \in S$ iff $C(d) \notin S$;
(3) exactly one of $Axy, Oxy \in S$;
(4) exactly one of $Ixy, Exy \in S$;
(5) at least one of $Ixy, Oxy \in S$;
(6) at most one of $Axy, Exy \in S$.

[17] Aristotle showed that, out of the following rules, the rules (C3), (PS3), (PS4) are redundant; they can be considered as derived rules (see Boger, 1998; Glashoff, 2005).

DEFINITION 2.4. (Martin, 1997) *S is a saturated maximally consistent set iff S is maximally consistent and*

(1) $Ixy \in S$ iff, for some z, Azx, $Azy \in S$;
(2) $Oxy \in S$ iff, for some z, Azx, $Ezy \in S$.

LEMMA 2.5. (Martin, 1997) *Every consistent set is extensible to a saturated maximally consistent set.*

§3. An intensional semantics. Let us go back to the notion of intensional interpretation of (1.2.2) which we will now define precisely.

Let M denote a nonempty set, let $\Omega \subseteq 2^M$ denote a nonempty subset of the power set of M, and let $\mathcal{O} = \Omega \times \Omega$.

DEFINITION 3.1.[18] *We will call a pair* $(\mathcal{O}, i)$ *consisting of* $\mathcal{O}$ *and a function*

$$i \quad : \quad T \cup \mathcal{L} \rightarrow \mathcal{O} \cup \{true, false\}$$

an intensional interpretation of the Aristotelian syntax, iff

(1) If $x \in T$, $i(x) = (s(x), \sigma(x)) \in \Omega \times \Omega$ and $s(x) \neq \emptyset$, $s(x) \cap \sigma(x) = \emptyset$
(2) If $d \in \mathcal{L}$, then
 (a) if d is some Axy, then $i(d) = true$ *iff* $s(y) \subseteq s(x)$ *and* $\sigma(y) \subseteq \sigma(x)$;
 (b) if d is some Exy, then $i(d) = true$ *iff* $s(x) \cap \sigma(y) \neq \emptyset$ *or* $s(y) \cap \sigma(x) \neq \emptyset$;
 (c) if d is some Ixy, then $i(d) = true$ *iff* $s(x) \cap \sigma(y) = \emptyset$ *and* $s(y) \cap \sigma(x) = \emptyset$;
 (d) if d is some Oxy, then $i(d) = true$ *iff* $s(y) \nsubseteq s(x)$ *or* $\sigma(y) \nsubseteq \sigma(x)$.

A *(syllogistic) intensional semantics* consists of a set of intensional interpretations $\mathcal{F} = \{(\mathcal{O}, i)\}$(for varying O as well as i). It is useful to have at hand the following standard notation:

DEFINITION 3.2. *Let* P *denote a set of sentences of* $\mathcal{L}$, *and let* $\mathcal{F}$ *be a given intensional semantics.*

- f =$(\mathcal{O}, i)$ satisfies P *iff* for all $d \in P$, $i(d) = true$;
- P is satisfiable in $\mathcal{F}$ *iff*, for some $f = (\mathcal{O}, i) \in \mathcal{F}$, i satisfies P;
- P is unassailable in $\mathcal{F}$ *iff*, for any $f = (\mathcal{O}, i) \in \mathcal{F}$, there is some $d \in P$, $i(d) = true$;
- $P \models_{\mathcal{F}} d$ *iff*, for all $f = (\mathcal{O}, i) \in \mathcal{F}$, f satisfies P only if $i(d) = true$;
- $\models_{\mathcal{F}} d$ *iff* $\emptyset \models_{\mathcal{F}} d$.

As usual, we write $P \models d$ *iff* $P \models_{\mathcal{F}} d$ for *all* intensional semantics $\mathcal{F}$. It is now easy to confirm the following *semantic version* of some of the items of Lemma 2.3:

LEMMA 3.3. *Using the notation of Section 2,*

1. $Axy \models Ixy$ *and* $Exy \models Oxy$;
2. $\{Axy, Exy\}$ *is not satisfiable and* $\{Ixy, Oxy\}$ *is unassailable.*

[18] This definition is a generalization of Leibniz' concept of (pairs of) characteristic numbers, see Glashoff (2002).

Proof.

1. Let us assume $i(Axy) = true$ for some intensional interpretation $f = (\mathcal{O}, i)$. Then, by Definition 3.1 (2a), $s(x) \subseteq s(y)$ and $\sigma(x) \subseteq \sigma(y)$; in addition (by Definition 3.1 (1)), $s(y) \cap \sigma(y) = \emptyset$. This implies $s(x) \cap \sigma(y) = \emptyset$ as well as $s(y) \cap \sigma(x) = \emptyset$; hence, according to Definition 3.1 (2c), $i\ (Ixy) = true$. The proof of the second part is similar.
2. If $f = (\mathcal{O}, i)$ is an interpretation such that $i(Axy) = true$, then, as we have just seen in the proof of the first part of the lemma, $i(Ixy) = true$. But, by inspection of Definition 3.1 (2b, 2c), this implies $i(Exy) = false$. This proves the first part. The second part follows in the same way. □

LEMMA 3.4. *Let P denote an arbitrary maximally consistent saturated set. Then there is an intensional interpretation $f = (\mathcal{O}, i)$ such that $i(d) = true$ iff $d \in P$.*

Proof. Let $\Omega = 2^T$, and let us define, for $x \in T$,

$$\begin{aligned} i(x) &= (\{x\} \cup \{y \in T \mid Axy \in P\}, \{y \in T \mid Exy \in P\}) \\ &= (s(x), \sigma(x)) \,. \end{aligned}$$

Let us first show that this defines indeed an intensional interpretation according to Definition 3.1. For this, we have only to prove $s(x) \neq \emptyset$ and $s(x) \cap \sigma(x) = \emptyset$. (a): As $x \in s(x)$ for any $x \in T$, $s(x) \neq \emptyset$. (b) Let $z \in T$ such that $z \in s(x) \cap \sigma(x)$. Then, Axz and Exz are in P, which is impossible by the consistency of P.

We then prove that, for any $d = Uxy$, $U \in \{A, I, E, O\}$, $x, y \in T$, it holds true that $d \in P$ *iff* $i(d) = true$.

Case 1: $d = Axy$. "$\Rightarrow$" : First we show that $d \in P$ implies $s(y) \subseteq s(x)$ *and* $\sigma(y) \subseteq \sigma(x)$. For any $z \in s(y)$, we have $Ayz \in P$ which, together with $d = Axy \in P$, results in $Axz \in P$, by (PS1), thus $z \in s(x)$. Hence $s(y) \subseteq s(x)$. For any $z \in \sigma(y)$, $Eyz \in P$. Together with $Axy \in P$, this implies (by means of (PS2)) $Exz \in P$, hence $z \in \sigma(x)$. Therefore, $\sigma(y) \subseteq \sigma(x)$. "$\Leftarrow$" : Let us assume that $s(y) \subseteq s(x)$ and $\sigma(y) \subseteq \sigma(x)$, but (*reductio*) $Axy \notin P$. Since P is maximal, $Oxy \in P$. By saturation, there is a $z \in T$ such that $Azx, Ezy \in P$. Hence, $z \in \sigma(y)$ and $z \in \sigma(x)$. Hence, $Ezx \in P$ which is not possible, since $Azx \in P$ has been assumed to hold.

Case 2: $d = Exy$. "$\Rightarrow$" : As $Exy \in P$ implies $Eyx \in P$, it follows that $x \in \sigma(y)$, implying $s(x) \cap \sigma(y) \neq \emptyset$ because of $x \in s(x)$. According to the definition, $i(d) = true$. "$\Leftarrow$" : In the first case, assume $s(x) \cap \sigma(y) \neq \emptyset$; hence there is a $z \in T$ such that $Axz \in P$ and $Eyz \in P$ implying $Exy \in P$ by means of (PS2). Second: $s(y) \cap \sigma(x) \neq \emptyset$ implies $Ayz, Exz \in P$ for some $z \in P$, hence again $Exy \in P$.

Case 3: $d = Ixy$. "$\Rightarrow$" : By saturation, there is a $z \in T$ such that $Azx, Azy \in P$. Hence $s(x) \cup s(y) \subseteq s(z)$ and $\sigma(x) \cup \sigma(y) \subseteq \sigma(z)$ (see Case 1 of this proof). By definition of $s(z)$ and $\sigma(z)$, $s(z) \cap \sigma(z) = \emptyset$. Hence $s(x) \cap \sigma(y) = \emptyset$ and $s(y) \cap \sigma(x) = \emptyset$, implying $i(d) = true$. "$\Leftarrow$" : Let us assume $Ixy \notin P$; hence $Exy \in P$ by Lemma 2.3(4). Case 2 of this proof shows that $i(Exy) = true$, hence $i(Ixy) = false$, which proves the assertion.

Case 4: $d = Oxy$. "$\Rightarrow$" : By saturation, for some $z \in T$, $Azx, Ezy \in P$. Hence, by $Azx \in P$, $s(x) \subseteq s(z), \sigma(x) \subseteq \sigma(z)$. Let us assume (*reductio*) that $s(y) \subseteq s(x)$ and $\sigma(y) \subseteq \sigma(x)$, then $s(y) \subseteq s(z)$ and $\sigma(y) \subseteq \sigma(z)$. Hence $Ayz \in P$, contradicting $Ezy \in P$. "$\Leftarrow$" : Let us assume $Oxy \notin P$, hence $Axy \in P$ by Lemma 2.3(3). Case 1 of this proof (" $\Rightarrow$ ") shows that $i(Axy) = true$, hence $i(Oxy) = false$, which proves the assertion. □

THEOREM 3.5. *(Intensional correctness theorem): $P \vDash d$ iff $P \vdash d$.*

Proof.

1. *Soundness* (" $\Leftarrow$ ") is proven by induction, checking the rules of the system—conversions and syllogisms—one after another. We will skip this point.
2. *Completeness* (" $\Rightarrow$ ") will now be proven utilizing a standard HENKIN-style of argumentation. Let $P \vDash d$. Then, $P \cup \{C(d)\}$ is inconsistent. If not, there would exist by the foregoing lemma, an intensional interpretation $f = (\mathcal{O}, i)$ such that f satisfies P and $C(d)$. Because of $i(C(d)) = true$ *iff* $i(d) = false$, the existence of i would contradict $P \vDash d$. The inconsistency of $P \cup \{C(d)\}$ implies that there are finite subsets $Q, Q' \subset P$ and a $g \in \mathcal{L}$ such that $Q \cup \{C(d)\} \vdash g$ and $Q' \cup \{C(d)\} \vdash C(g)$. Hence $Q \cup Q' \vdash d$ by indirect deduction, Definition 2.2(2). Since $Q \cup Q' \subseteq P$, $P \vdash d$. □

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