ARISTOTELIAN SYNTAX FROM A COMPUTATIONAL-COMBINATORIAL POINT OF VIEW

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ABSTRACT. This paper translates Aristotelian logic into the sphere of computational-combinatorical research methods. The task is accomplished by formalising ARISTOTLE'S logical system in terms of rule-based reduction relations on a suitable basic set, which allow us to apply standard concepts of the theory of such structures (NEWMAN lemma) to the ancient logical system. In this way we are able to reproduce ARISTOTLE'S method of deriving syllogisms within a precisely defined formal environment, and we can analyse the structure of the set of syllogistic rules by means of a computer program. Thus we show that Aristotle's syllogistic logic is a formal system of its own, which can be modelled independently of predicate logic and set theory. Our research is very much in the spirit of SMILEY'S and CORCORAN'S modelling of Aristotelian logic as a deductional system, and we aim to staying close to ARISTOTLE'S own term–logical concepts.

INTRODUCTION

Unlike several decades ago, Aristotelian logic meets with growing interest today. Not only philosophers, but also specialists in information and communication theory employ ideas which can be explicitly traced back to ARISTOTLE'S work on categories and syllogisms. Let us mention the project of the *Semantic Web*, where formal ontologies belong to the main building blocks of a gigantic world-wide information system. This projected system even outstrips the futuristic plans of LEIBNIZ and his *characteristica universalis*[1].

Independently of these rather recent developments, there has been a renewed interest in matters of formalization of Aristotelian logic by a small group of logicians, philosophers and philologists. The publication of ŁUKASIEWICZ' book [10] on Aristotelian logic about half a century ago and, roughly 20 years later, the work of CORCORAN [4] and SMILEY [15], opened a new dimension for the research on Aristotelian term logic. These authors extended the traditional research topics into questions of soundness and completeness of the ancient system. In addition, the latter authors try to conform as much as possible to the original texts, and their work is considered as a genuine symbolic representation of ARISTOTLE'S syllogistic. Furthermore, in contrast to ŁUKASIEWICZ' writings, the theories of CORCORAN and SMILEY link smoothly to the traditional syllogistic, which has been the predominant logic since ancient times, extending over approximately 2000 years up to the beginning of modern symbolic logic in the middle of the 19th century.

This paper conforms with CORCORAN'S and SMILEY'S work by adopting a strictly formal approach to Aristotelian logic. Our method is clearly contradistinctive to the attempts of early modern logicians, who misread Aristotelian logic as a simple or even faulty special case of predicate or class logic. Unlike the aforementioned authors, however, we aim to position Aristotelian logic within the sphere of computational-combinatorical research methods. This is to provide a solid formal fundament for further computer–aided research into this ancient system of formal logic[2].

In the following, we will deal solely with the *syntax* of the syllogistic logic of ARISTOTLE [3]. To be more specific: We will explore the inner structure of the set of Aristotelian syllogisms – a

[1]An introduction into the modern theory of formal ontology and some of its relations to ancient logic is SOWA [18]. - The *Semantic Web* is a project of the W3C (World Wide Web Consortium). It aims at the extension of the current web in that a well defined *meaning* will be assigned to each peace of information. The whole project is too big and too ambitious to be described here, be it only roughly. Cf. *http://www.w3.org* for detailled information.

[2]It goes without saying that we will have to bridge abysses in trying to provide our methods and results in a form which make them accessible for those interested in the interpretation of classical Aristotelian logic as well as for those concerned with the properties of the formal system.

[3]We plan to treat semantic matters in a subsequent paper.

research topic which, in some of its results but not in its method, is in the spirit of parts of Paul THOM's book [20] [4].

Our approach is new insofar as we put ARISTOTLE'S logic into a framework which we consider to be the proper formal context for a computational-combinatorical treatment of this ancient logic: Syllogisms are modelled as *reduction relations* on sets of categorical propositions. This approach allows us to give a very precise meaning to all formal aspects of the syllogistic theory and determines its implementation by means of computer programs.

The structure of this paper is as follows. In the first Section we will present a semi-formal introduction to the basic syntax of Aristotelian logic, and we will show how to formalise this logical system in terms of rule-based reduction relations on a suitable basic set. This will be put into a precise formal context in Section 2.

While Section 1 and 2 deal with the basic *first* syntactical level, Section 3 of our paper is devoted to the *second* syntactical level of Aristotelian logic, which is the formal theory of direct and indirect *derivation* of syllogisms. In terms of the vocabulary of formal logic, *derivation* of syllogisms denotes the generation of *derived rules*. The fourth and final Section contains some results of our computational exploration of the syllogistic system. Utilising a computer program[5] based on the reduction relation concept, we establish some theorems on the inner structure of the syllogistic system. As a conceptual frame we take the notion of *syllogistic bases*, and this will allow us to compare our conclusions with the results of previous works of ARISTOTLE, LEIBNIZ, ŁUKASIEWICZ, CORCORAN, SMILEY, and THOM.

Our imbedding of Aristotelian logic into the framework of rule-based reduction systems, well known from other parts of formal logic (λ–calculus, automatic theorem proving, computer algebra systems), and our employment of computer-based methods to explore this ancient logic serves as a further affirmation of the fact that Aristotelian logic should not and cannot be considered as a simple and incomplete forerunner of predicate logic or even set theory. Hopefully, our paper may help in understanding ŁUKASIEWICZ' ([10][6], p.131) dictum that

> "The syllogistic of Aristotle is a system the exactness of which surpasses even the exactness of a mathematical theory, and this is its everlasting merit."

Dedication. The only work I know of, which employs computational methods as a tool for the study of Aristotelian logic is Claus BRILLOWSKI's doctoral dissertation ([3]). It was my friend Claus who roused my interest in Aristotelian logic, and this paper is dedicated to him, who, having tried to draw my attention to ARISTOTLE for many years, has finally succeeded.

1. ARISTOTELIAN TERM LOGIC: DEDUCTION

In presenting a minimal formal background for research into the Aristotelian sollogistics, it is difficult to avoid taking position in a, by now classical, dispute on the 'correct' way of formalising Aristotelian logic. George BOGER ([1], [2]) named the three main lines of interpretative trends regarding the Prior Analyics *traditionalists, axiomaticists*, and *deductionists.* ŁUKASIEWICZ is the most prominent *axiomatist,* CORCORAN and SMILEY are, in BOGER's terminology, *deductionists.* The *traditional* way of looking at ARISTOTLE'S syllogisms as a system of rules, applicable to sets of categorical syllogisms, is very similar to the deductionist view. Thus it is strongly related to our way of interpreting ARISTOTLE, as we do not conceal our strict adherence to the deductionist view.

In our approach to Aristotelian logic we consider two different levels of syntactical formal reasoning, which may be described as follows:

(1) *The basic level*: At this level, there are the usual definitions of terms, propositions, and rules of deductions. This is the *core logic* of the Aristotelian system.

[4]There are differences to THOM's work which will be discussed later on after we have presented our method.

[5]All results can, in principle and without theoretical problems, be reproduced by hand calculation. While this is certainly important for checking basic results, it is definitely too time consuming for all the results of Section 3.

[6]ŁUKASIEWICZ' book is the first modern presentation of a term logic - a task LEIBNIZ had tried to accomplish some 250 years before. ŁUKASIEWICZ was the first prominent representative of modern symbolic logic who realised, in explicit opposition to RUSSEL, that Aristotle's logic could *not* be classified as a mere special case of predicate logic. Today we know that, despite his merit with respect to a modern interpretation of Aristotelian logic, ŁUKASIEWICZ overshot the mark in taking propositional logic as the basis for ARISTOTLE'S syllogistic.

TABLE 1. Aristotelian propositions

Formal	**extensional**	**intensional**
Axy	All x are y	y applies to all x
Exy	No x is y	y applies to no x
Ixy	Some x is y	y applies to some x
Oxy	Some x are not y	y does not apply to all x

(2) *The rule level*: This is the level in which the rules of the system are the objects of research. As the main rules of the basic level are syllogisms, this second level corresponds to the classical theory of *derivation of syllogisms* by direct and indirect methods[7]. This part is the *theory of derived rules* which concerns different methods of producing new valid rules of the deductive system by using the basic given ones.

Most classical interpretations of Aristotelian logic blur the difference between these two levels or do not even allow for such a distinction of levels[8]. This does not imply that the results obtained by these methods are incorrect. Thus, for example, ŁUKASIEWICZ's interpretation, which does not discriminate between the two levels[9], may be wrong from a historical and philological standpoint. Nevertheless, his findings concerning the structure of the syllogistic system are by no means worthless from a formal point of view: The important results gained by this historically inappropriate interpretation can be easily translated into the formalism of, let us say, a calculus of natural deduction.[10] The same comment applies to THOM's book which uses an approach very similar to that of ŁUKASIEWICZ.

1.1. **The basic level.** In order to fully specify the basic level of Aristotelian logic, we first define the syntax of the *well formed formulas* of his logic:

Definition 1.1. (well formed formulas)

(1) We are given an infinite set T of (nonlogical) constants $t_1, t_2, t_3, ...$which play the role of *categorical terms* (e.g. 'man', animal', 'living being', ...). For the sake of simplicity, we also use the symbols $x, y, z, ...$for elements of T.
(2) There are four logical constants, $A, E, I$and O, sometimes called *Aristotelian quantors.*
(3) A *proposition* is a string consisting of one logical constant attached to to a string of two different nonlogical constants: e.g. $Axy, Eyx, ...$ We will abbreviate propositions by *p, q, r, ...*

In Axy we formalise the verbal expression 'All x are y' or, as ARISTOTLE put it, 'y applies to the whole x'. One could say that the first of these expressions refers to an *extensional* interpretation of the logical constant A, whilst the second one has an *intensional* flavour. ARISTOTLE used both phrases. Table 1 gives the usual extensional as well as the corresponding intensional expressions.

The first syntactical level of Aristotelian logic may be described in short by the following statements:

[7]We use the term 'derivation' of syllogisms instead of the more traditional notation 'reduction' in order to prevent confusion of names, because 'reduction relations' are central in our formal setting of the *basic–level* syntax of Aristotelian logic. In standard mathematical terminology, a reduction relation is simply a strict antisymmetric relation, and this notion has absolutely nothing in common with the classical term 'reduction', being often used for the Aristotelian process of the deduction of derived rules. See BOGER ([1], [2]) for a thorough investigation of the terms used by ARISTOTLE. Our terminology 'derivation of syllogisms' has the advantage that it goes well together with the concept of derived rules.

[8]In [2], BOGER precisely distinguishes these two levels: Our first, basic level corresponds to his 'deduction system of Aristotle's underlying logic' (p. 166, Table 10). Our *derivations* within the second syntactic level corresponds to BOGERS 'metalogical deductions for panvalid patterns' (p. 183 ff).

[9]In ŁUKASIEWICZ's approach, syllogisms belong to the basic objects of his logical system: they are just conditional propositions which can be manipulated by means of the rules of his system (which include the rules of the propositional calculus). Thus, in his system, there is no need for a 'rule level'. CORCORAN and SMILEY formulate the 'basic level' by means of a system of natural deduction. Within these systems, both levels are present, and derivation of syllogisms, in the framework of this setting, is a special method of deduction of derived rules.

[10]EBBINGHAUS, [5], who was the first to formalise Aristotelian logic in a strict deductionist manner, showed the formal equivalence of his system with that of ŁUKASIEWICZ.

- ARISTOTLE'S deductive system allows one to deduce, from a given set α of propositions, by means of certain well defined rules, other propositions which are different from the given ones.
- This point of view has the immediate consequence that the structure underlying the basic language level does not consist of propositions, but of *sets* of propositions.
- This in turn implies that the rules of the system (syllogisms and immediate rules) operate on *sets of propositions* and have to be formulated accordingly: A syllogism takes a set α of propositions as input and - if the syllogism *applies* - produces an output α' which is again a set of propositions, possessing exactly one proposition more than the input set α.

This way of looking at Aristotelian logic is, of course, due to our aim of modelling Aristotelian logic by means of a rule - based computer program. Nevertheless, it is equivalent to the traditional way of regarding syllogisms as rules, and also to CORCORAN'S and SMILEY'S way of modelling Aristotelian logic in terms of *a system of natural deduction*. The rules of this deductive system are taken from the logical papers of ARISTOTLE and his successors as has been convincingly shown by the authors mentioned above. It is important to stress that this approach is also very different from the attempts of early modern logicians who, unsuccessfully, tried to incorporate Aristotelian logic into standard predicate logic[11], as well as from ŁUKASIEWICZ' work which employs propositional logic.

1.2. **Aristotelian deduction.** The rules of deduction of the logical system considered here are taken from ARISTOTLE'S work; one of the famous rules is the so called *E-conversion* which we will use as a paradigm for the class of *immediate* rules of the system. Within our framework, this rule may be made precise by the following description (which, by the way, can easily be transformed into a computer program in whichever computer language):

Definition 1.2. *Rule of E-conversion:* Let $\alpha = \{p, q, ...\}$ be a given set of propositions. If α contains the proposition *Exy* for some *x* and *y*, but *not* the proposition *Eyx*, then replace α by α' which has exactly one proposition more then α, namely, the proposition *Eyx*:

$$\alpha = \{..., Exy, ...\} \rightarrow \alpha' = \{..., Exy, Eyx, ...\}.$$

We will also make use of a short hand notation for this rule, which we will denote by

$$Exy \rightarrow Eyx$$

It is important to understand that we use this notation as an *abbreviation* for the rule defined precisely above.

We want to stress the following consequences of our definition (this will also apply to all other rules we are going to encounter):

Remark 1.1. If, in addition to each proposition *Exy* contained in α, α contains also *Eyx*, then, according to our definition, the rule $Exy \rightarrow Eyx$ *cannot be applied!* Or, formulating it the other way round: *if* the rule *can* be applied, it will definitely enlarge the set of propositions to which it is applied. This, of course, is just a matter of arbitrarily fixing the use of the expression 'application of a rule', but this assessment will turn out to be appropriate later on.

Remark 1.2. For a given set α of propositions, there may be different possibilities to apply the rule. E.g., if $\alpha = \{Auv, Exy, Ezt, Iyz\}$, the rule can be applied in two different ways, thus leading to the same final set[12] via two different paths:

$$\begin{array}{rl} & \{Auv, Exy, Est, Iyz\} \rightarrow \{Auv, Exy, Est, Eyx, Iyz\} \\ & \quad \rightarrow \{Auv, Exy, Est, Eyx, Ets, Iyz\} \\ or & \{Auv, Exy, Est, Iyz\} \rightarrow \{Auv, Exy, Est, Ets, Iyz\} \\ & \quad \rightarrow \{Auv, Exy, Eyx, Est, Ets, Iyz\} \end{array}$$

Thus, a rule like the one of E-conversion is *not a function* which is defined on the set of subsets of the set of propositions, but rather a two-place *relation* on that set of subsets.

There are, in total, six classical *immediate* rules to which everything said above also applies. Therefore we will not formulate them explicitly as rules like we did it with respect to E-conversion but just give their shorthand notation in the following table.

[11] c.f. Strawson, [19], p. 152 - 194.

[12] The elements of a *set* are not ordered!

TABLE 2. *Immediate rules* of conversion and subalternation

Classical name	**Rule**	**acronym**
E-conversion	$Exy \to Eyx$	E-con
I-conversion	$Ixy \to Iyx$	I-con
(partial) A-conversion	$Axy \to Iyx$	A-pcon
(partial) E-conversion	$Exy \to Oyx$	E-pcon
A-subalternation	$Axy \to Ixy$	A-sub
E-subalternation	$Exy \to Oxy$	E-sub

Most modern treatises on Aristotelian logic contain, at this point, a caveat concerning the rule of partial A-conversion. For example this one, cited from SMITH [17]:

> " From a modern standpoint, the third [i.e., the rule of partial A-conversion; K.G.] is sometimes regarded with suspicion... In fact this simply points up something about Aristotle's system: Aristotle in effect supposes that all terms in syllogisms are not empty."

This remark, being representative for almost all modern comments on ARISTOTLE'S system, is problematic for the following reason. First, a *term* can neither be empty nor non–empty. Second, the problem of 'empty terms' arises only in the context of the *interpretation* of terms as sets of individuals. *If* one performs such an interpretation, *then* one has to restrict oneself to nonempty sets. ARISTOTLE, however, never explicitly proposed this kind of interpretation. Thus this whole question comes into existence only if we discuss the *semantics* of Aristotelian logic. *If* we then prefer to interpret Aristotelian terms as sets, then, of course, these sets have to be nonempty. However, it is not true that ARISTOTLE had this kind of extensional interpretation in mind! Already ŁUKASIEWICZ warned his fellow logicians, first of all RUSSELL, of this danger of misinterpreting Aristotle ([10], p. 219). Let us repeat that, within the present context, we do not care about special interpretations of Aristotelian logic but remain strictly on the syntactical level – on this level, where terms stay uninterpreted, there is absolutely no problem with any of the immediate rules.

Example 1.1. *Let $\alpha = \{Axy, Ayz, Azy\}$. Using the conversion- and subalternation-rules, we obtain the following two different sequences of sets of propositions, leading each to the same final set:*

$$\alpha = \{Axy\} \overset{A-sub}{\longrightarrow} \alpha^1 = \{Axy, Ixy\} \overset{I-con}{\longrightarrow} \alpha'' = \{Axy, Ixy, Iyx\}$$

$$\alpha = \{Axy\} \overset{A-pcon}{\longrightarrow} \alpha^2 = \{Axy, Iyx\} \overset{I-con}{\longrightarrow} \alpha'' = \{Axy, Iyx, Ixy\}$$

1.3. **Syllogistic rules.** Syllogistic rules are special rules which utilise *two* different given terms in order to produce a new term. Again, we start with an example, the well known and central *Barbara* syllogistic rule. As in the case of immediate rules (cf. the preceding subsection), there is an abbreviating notation for syllogistic rules which, in the case of *Barbara*, reads

$$Axy, Ayz \to Axz$$

The following propositions define precisely the meaning of this abbreviation:

Definition 1.3. *Rule of the Barbara syllogism:* Let $\alpha = \{p, q, ...\}$ be a given set of propositions. If α contains, for some three different terms x, y and z, the propositions Axy and Ayz, but not (yet) the proposition Axz, then replace α by the set α' which has exactly one proposition more than α, namely, Axz:

$$\alpha = \{...., Axy, ..., Ayz, ...\} \longrightarrow \alpha' = \{..., Axy, ..., Ayx, Axz, ...\}$$

The other syllogistic rules differ from *Barbara* in that they also contain other Aristotelian constants, not just A. There are exactly four different types of rules (corresponding to the four classical *figures*). They differ in the *distribution* of the 'middle term' y and its placement in relation to the 'outer terms' x and z [13]. In the following table, we let U, V, and W stand for the logical constants A, E, I and O.

[13]We will not participate in discussions about the classification of syllogisms in terms of figures. While this may be an entertaining discussion from a historical standpoint, it is totally useless from a formal point of view. Since

TABLE 3. The syllogistic figures

Figure	1. premise	2. premise	conclusion
1	Uxy	Vyz	Wxz
2	Uxy	Vzy	Wxz
3	Uyx	Vyz	Wxz
4	Uyx	Vzy	Wxz

By varying *U, V,* and *W,* one obtains, for each figure, 4*4*4=64 syllogistic rules. This makes 4*64=256 syllogistic rules in total. Each of these rules has both a shorthand representation as well as a detailed definition like *Barbara*, (see Definition 1.3). Each rule will also be given a unique label of the form (n, U, V, W), where n is the number of the figure ($n = 1, 2, 3$, or 4), and U, V, W are each one of the constants A, E, I, and O. Thus, *Barbara* has the label $(1, A, A, A)$, and the second most famous syllogistic rule, *Celarent*, has the label $(1, A, E, E)$[14].

1.4. **The syllogistic system.** It is important not to confuse the 256 syllogistic rules with the term 'syllogism' (gr. *sullogismos*) that appears in ARISTOTLE'S work: ARISTOTLE'S syllogisms are syllogistic rules, but not vice versa. The very core of Aristotelian logic is a certain subset of the set of 256 syllogistic rules; this subset constitutes the set of *syllogisms* or *valid syllogistic rules* [15]. Aristotle himself 'certified' 14 of the syllogistic rules as syllogisms. Later logicians found that, within the system constituted by ARISTOTLE, there are exactly 24 valid syllogistic rules, i.e. 24 syllogisms. A substantial part of Aristotelian logic deals with the justification of the difference between valid and invalid syllogistic rules, a subject which we will now step into.

First we present the list of 24 syllogistic rules - it is that special subset of all 256 rules commonly known as *syllogisms.* At this point in our survey we cannot justify the selection of just *this* subset, but this will soon be made up for.

1.5. **A rule set for Aristotelian logic.** Let us note that we are still occupied with the basic *first* syntactic level of Aristotelian logic. Up to now we have not exactly specified which rules of derivation will be taken as *the* rules of the system. From a formal standpoint, there is a lot of arbitrariness at this point. Thus we conform to what Aristotle himself presented as the basis of his system. This set of basic rules is composed of some of the rules of conversion and subalternation as well as the first four syllogistic rules of the first figure :

In his *Prior Analytics*, ARISTOTLE first assumes these rules as being obviously given, then he elaborates the consequences of his assumptions. By reasoning with the rules 1. - 6. of Table 5, he is able to show that there are 10 more valid syllogistic rules:

- Cesare $(2, A, E, E)$, Camestres $(2, E, A, E)$, Festino $(2, I, E, O)$, Baroco $(2, O, A, O)$,
- Darapti $(3, A, A, I)$, Disamis $(3, A, I, I)$, Datisi $(3, I, A, I)$, Felapton $(3, A, E, O)$, Bocardo $(3, A, O, O)$, Ferison $(3, I, E, O)$.

Ignoring, for the moment, the *method* by which ARISTOTLE arrived at this result about his 14 syllogisms, we may take this system of 3 'immediate' together with 14 syllogistic rules - these are

LEIBNIZ' works on syllogisms, about 1700 A.D., there has not been *any* contribution to the theory of syllogistic figures which is both new and meaningful.

[14]It is convenient, for some purposes, to introduce a consecutive numbering system for syllogistic rules. Each rule gets a unique number, the *index* of the rule, within the range 1 to 256 as follows:

- Take the label (n, U, V, W) of the rule;
- replace A, E, I, O with the numbers $1, 2, 3$, and 4, respectively.
- Let (n, i, j, k) be the resulting quadruple of numbers, each of them between 1 and 4.
- Subtract 1 from each of the numbers n, i, j, and k. Let the result be (n_1, i_1, j_1, k_1).
- Regard $n_1 i_1 j_1 k_1$ as the base-4-representation of a decimal number N;
- compute the index $N = k_1 + 4j_1 + 16i_1 + 64n_1 + 1$.

As an example, let us compute the index of *Celarent* which has the label $(1, A, E, E)$: Replacing A and E by 1 and 2, we obtain $(1, 1, 2, 2)$; subtracting 1 from all four numbers, we obtain $(0, 0, 1, 1)$. Thus, the index of *Celarent* is $N = 1 + 1{*}4 + 1 = 6$.

[15]Using BOGER's terminology in [1], who refers to PATZIG [13]: Our 256 syllogistic rules are *argument patterns*, and a syllogism is a *concludent pattern*.

TABLE 4. The Syllogisms

No.	Name	Label	Index	Aristotelian
1	Barbara	(1,A,A,A)	1	*
2	Celarent	(1,A,E,E)	6	*
3	Darii	(1,I,A,I)	35	*
4	Ferio	(1,I,E,O)	40	*
5	Barbari	(1,A,A,I)	3	
6	Celaront	(1,A,E,O)	8	
7	Cesare	(2,A,E,E)	70	*
8	Camestres	(2,E,A,E)	82	*
9	Festino	(2,I,E,O)	104	*
10	Baroco	(2,O,A,O)	116	*
11	Cesaro	(2,A,E,O)	72	
12	Camestrop	(2,E,A,O)	84	
13	Darapti	(3,A,A,I)	131	*
14	Disamis	(3,A,I,I)	139	*
15	Datisi	(3,I,A,I)	163	*
16	Felapton	(3,A,E,O)	136	*
17	Bocardo	(3,A,O,O)	144	*
18	Ferison	(3,I,E,O)	168	*
19	Bamalip	(4,A,A,I)	195	
20	Camenes	(4,E,A,E)	210	
21	Dimatis	(4,A,I,I)	203	
22	Fesapo	(4,A,E,O)	200	
23	Fresison	(4,I,E,O)	232	
24	Camenop	(4,E,A,O)	212	

TABLE 5. ARISTOTLE's first system

1.	E-conversion	E-con
2.	partial A-conversion	A-pcon
3.	I-conversion	I-con
3	Syllog. rule (1,A,A,A)	Barbara
4.	Syllog. rule (1,A,E,A*)*	Celarent
5.	Syllog. rule (1,I,IA)	Darii
6.	Syllog. rule (1,I,E,O*)*	Ferio

the starred ones in Table 4 - as a *definition* of his logical system, i.e. as the definition of the basic first syntactical level of his logic.[16]

Definition 1.4. The basic level of Aristotle's logic is constituted by

(1) *Well formed formulas:* Sets of categorical propositions.
(2) *Rules of deduction:* E-conversion, partial A-conversion, I-conversion plus the 14 Aristotelian syllogisms (the starred ones in Table 4)
(3) *Deduction method:* Direct deduction (simple rule - application).

Whilst this formal setting conforms closely to what Aristotle did within his system of the *Prior Analytics*, there are two questions arising immediately:

- *1st question*: How does this set of 14 syllogisms rules relate to the set of 24 syllogistic rules given above in Table 4? This is the oft-discussed question of whether or why the

[16] We will come back to the 'basis' of Table 5 in Section 4. There we will discuss the precise meaning of the term 'syllogistic basis', and we will show how the basis of Table 5 fits into a general theory of bases of Aristotelian logic.

set of 14 syllogisms is 'too small'. Like almost all formal systems during the history of Aristotelian logic, the answer *our* formal system will produce is: utilising the methods given by ARISTOTLE himself, 10 more syllogisms can be shown to be valid, thus leading to those 24 syllogisms which are have been well known since medieval times.
- *2nd question*: Do we really need 14 syllogisms? We will show that *exactly six* of the 14 syllogisms suffice[17], in that they generate the same set of propositions as the full set of 14 (or, of course, 24) syllogisms.

Let us stress that, so far, in our presentation of the basic syntactic level of ARISTOTLE'S system, the method of *indirect* reasoning or reasoning by contradiction has not been included: Even if this method plays an important role in ARISTOTLE'S work, it is not indispensible or essential *at the basic syntactic level*. We will use the method of indirect reasoning in Section 3 when we discuss the second syntactical level, where syllogistic rules are the *objects* of formal reasoning. This is exactly the level at which ARISTOTLE used indirect reasoning, too.

In the following section we will construct a precise formal framework for the results which we indicated above. This will lead to a formulation of Aristotelian logic which is especially well suited to computer aided research into the structure of ARISTOTLE'S theory. We will employ the theory of rule-based systems (c.f. [8]) into which Aristotelian logic will be shown to fit smoothly.

2. A FORMAL SETTING FOR ARISTOTELIAN LOGIC

This algorithm–oriented presentation is an alternative to the formalism of Aristotelian logic as a system of natural deduction which has been predominant in theoretical environments up to now[18]. First, we will show how to precisely formalise Aristotle's logic within the theoretical framework of reduction relations on a suitable set. This in turn will allow us to apply standard concepts of the theory of reduction relations to the logical system under consideration.

2.1. The formal setting.

Definition 2.1. The basic ingredients of our formalism are as follows:
- the set $T = \{t_1, t_2, ...\}$ of *terms*;
- the set $Q = \{A, E, I, O\}$ of Aristotelian *quantors*
- the set $\Pi = \{Uxy/U \in Q, x \in T, y \in T, x \neq y\}$ of Aristotelian *propositions*;
- the set $\Sigma = 2^{\Pi}$, i.e. the set of all sets of propositions.

Σ is the fundamental set of Aristotelian logic[19]. Σ is fundamental, because Aristotelian rules will always be applied to *sets of propositions*, i.e., to elements of Σ. Applying a rule to a set of propositions in this way generates a new set of propositions in a way which we will describe below. We proceed by interpreting the process of rule–application in terms of certain binary relations on the basic set Σ, our intention being the development of a formal structure which enables us to give a precise mathematical formulation of the process of applying syllogistic rules. This, of course, is the process which we have already defined, in an informal manner, in Section 1.

Fortunately we do not have to invent a new formalism, as the theory of *reduction relations* is well established and has been well applied within many theoretical environments and for many different purposes ([8]). This theory, now indispensible within computer science, is an offspring of the $\lambda-calculus$, which is one of the most famous and useful foundational theories of our computer age.

Our method is to regard the rules of Aristotelian logic – immediate as well as syllogistic rules – as binary relations on the set Σ: Any such relation connects a set α of propositions (the assumptions) to a set α' (the assumptions together with *one* of the possible conclusions). The deduction process may now be described by a sequence of transitions, starting with a given set of assumptions, each step of which is generated by a certain arbitrarily selected rule. This process will be carried through until no rule of the logical system can be applied anymore.

We will now specify these notions, and then we will show that, in case we start the process with a finite set of propositions, the method will always lead to a unique final set. This in turn will allow us to define a well defined closure operator Φ for Aristotelian logic: $\Phi : \Sigma \to \Sigma$ maps a set

[17] These six are the four syllogisms of the first figure plus *Baroco* $(2, O, A, O)$ and *Bocardo* $(3, A, O, O)$.
[18] Readers who are not interested in the particulars of the formal system may skip Section 2.
[19] Examples of elements of Σ are finite sets like $\{At_1t_2, Et_2t_3\}$ or infinite sets like $\{At_1t_2, At_2t_3, At_3t_4, ...\}$.

α of propositions (assumptions) to a set β which contains the assumptions α together with *all* its logical consequences.

Definition 2.2. Let r denote a *relation* on Σ: $r \in \Sigma \times \Sigma$. If, for some $\alpha, \beta \in \Sigma$, $(\alpha, \beta) \in r$, we write $\alpha \stackrel{r}{\longrightarrow} \beta$.

We now define a set of relations on Σ which constitute Aristotelian logic: These relations are exactly those which belong to the rules which we have described at length in Section 2. For example, r_{E-con} is that special relation on Σ which is defined as follows[20]:

Definition 2.3. Two sets $\alpha, \beta \in \Sigma$ are related by r_{E-con}, i.e., $(\alpha, \beta) \in r_{E-con}$ (or, equivalently, $\alpha \stackrel{r_{E-con}}{\longrightarrow} \beta$), if and only if there is, in α, a proposition Et_it_k such that

- Et_kt_i is not in α
- Et_kt_i is in β
- $\alpha \subset \beta$, and Et_kt_i is the *sole* element of β which is not in α.

Remark 2.1. This definition implies that, for any $(\alpha, \beta) \in r_{E-con}$, β possesses exactly one element more than α.

In this way, to any immediate rule of Table 2, there exists a corresponding relation on Σ. Analogously, to any one of the 256 syllogistic rules, there exists a corresponding relation on Σ: Each syllogistic rules relates two elements α and β of Σ, if and only if it is possible to apply the rule to the set α of propositions, producing as result the set β. In this case, β contains always exactly one element more than α.

Thus, we are able to define a finite set of relations on Σ, which we shall denote as follows:

- $r_1, r_2, ..., r_{24}$: These are the relations generated on Σ by the 24 syllogisms of Table 4;
- $r_{25}, ..., r_{30}$: These are the 6 relations belonging to the immediate rules of Table 2.

We now consider the *union* of all these relations:

Definition 2.4. Let r denote the relation $r_1 \cup r_2 \cup ... \cup r_{30}$; α and β are related by r if and only if, for some k, $1 \leq k \leq 30$, $(\alpha, \beta) \in r_k$ – or, written in a different manner, $\alpha \stackrel{r_k}{\longrightarrow} \beta$.

Now all ingredients of a formal and constructive environment of Aristotelian logic are present:

Definition 2.5. The constructive formal system of Aristotelian logic is given by

- The basic sets T (terms), Q (quantors), Π (propositions), and Σ (sets of propositions) as defined in Def. 2.1;
- the binary relation $r = r_1 \cup r_2 \cup ... \cup r_{30}$ defined in Def. 2.4 by means of the relations belonging to the 30 rules of Table 4 and Table 2.
- The process of deriving a new set of popositions from a given one: $\beta \in \Sigma$ is said to be a direct derivation from $\alpha \in \Sigma$, written as $\alpha \stackrel{r}{\longrightarrow} \beta$, iff $(\alpha, \beta) \in r$.

Lemma 2.1. r is a reduction relation on Σ, i.e., a strict antisymmetric relation (satisfying $r \cap r^{-1} = \emptyset$).

Proof. It is not possible that, for two elements α, β of Σ, $(\alpha, \beta) \in r$ as well as$(\beta, \alpha) \in r$: Each relation r_k, $k = 1, ...30$, has the property that, from $\alpha \stackrel{r_k}{\longrightarrow} \beta$, it follows that β has exactly one element (one proposition) more than α . Thus it is not possible that *both* $\alpha \stackrel{r}{\longrightarrow} \beta$ and $\alpha \stackrel{r}{\longrightarrow} \beta$. □

Taken together, all these definitions now lead to a certain standard situation which allows us to put Aristotelian logic into the framework of reduction relations:

- Given a set Σ together with a reduction relation r on Σ.

Let us note that this a very natural formulation of the formal structure of Aristotelian logic: Within this setting, elements of Σ denote sets of Aristotelian propositions, and r represents the collection of syllogistic and immediate rules of the Aristotelian system.

The theory of reduction relations is a well investigated subject area within theoretical computer science, and we will profit from previous reseach in this area which originated from the theory of the $\lambda-$calculus.

The main questions in this field are the following:

[20] c.f. Table 5.

(1) *Existence of a normal form:* If one starts with an element $\alpha \in \Sigma$ and performs a sequence of reductions [21] $\alpha \longrightarrow \alpha_1 \longrightarrow \alpha_2....$, – under which conditions will this sequence of elements α_i stop? I.e., under which conditions does there exist a *normal form* β of α, such that $\alpha \longrightarrow \alpha_1 \longrightarrow \alpha_2.... \longrightarrow \beta$, where β is an element with the property that there does *not* exist any $\delta \in \Sigma$ with $\beta \longrightarrow \delta$?
(2) *Uniqueness of normal forms:* Is it possible that, for some initial element α, there exist *two different* normal forms, β and γ?

Let us translate these questions into the sphere of Aristotelian logic.

(1) Assume that we start with a set $\alpha = \{p, q, ...\}$ of Aristotelian propositions. We apply syllogisms as well as immediate rules as long as they are applicable, i.e., as long as, by application of *any* of the rules, a new proposition will be generated. Will this process always stop, ending with a maximal set of propositions which have been generated from the premise set α?
(2) If this process of applying rules to the set α of premises stops at β – is this final element β (a set of propositions, in our case) independent of the *order* in which the rules of the system have been applied during the process? Let us remember that not only are there different rules r_k hidden in r, but that, for each element ϑ and each rule r_k, there are in general many different $\xi' s$ with $\vartheta \xrightarrow{r_k} \xi$.

In the next subsection we will show that the existence of normal forms holds for any *finite* set of premises, and that this normal form is always uniquely determined. This fact will allow us to define a closure operator

$$\Phi : \Sigma \longrightarrow \Sigma$$

which maps finite sets of propositions α into their *Aristotelian closure* $\Phi(\alpha)$.

2.2. **The theory.** In this subsection we consider the special situation where the sets of propositions are finite. Thus, let Σ_f denote the set of all those elements of Σ which possess only finitely many propositions. Of course, $\Sigma_f = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup ....$ where each Σ_k consists of all elements of Σ possessing exactly k elements.

Lemma 2.2. On Σ_f, any sequence of reductions $\alpha \longrightarrow \alpha_1 \longrightarrow \alpha_2....$ terminates.

Proof. The statement follows from the fact that, for a given finite number of terms, there are only finitely many sets containing only propositions made up by these terms: Let $\alpha \in \Sigma_k$, then there are at most $t \leq 2k$ different terms within the propositions of α. Now the rules of the system do not enlarge the number of *terms* (however, of course, the number of propositions). Therefore, as there are $n \leq 4 * t(t-1)$ different propositions containing exactly two of these t different terms, there are at most $2^n - 1$ different subsets of propositions containing exactly these t terms. Thus, all possible outcomes of all rules applied to a set α with k premises are contained in a finite set with at most $2^{8k(2k-1)} - 1$ elements. As each application of a rule increases the number of elements of α by exactly one, the rules can only be applied a finite number of times. □

Example 2.1. *If, for instance, we consider an example with a premise set of $k = 2$ propositions, there are at most 2^{48} different sets of propositions containing only the maximal number of 4 terms contained in the premise set. This is, however, usually a crass overestimation of the number of steps required to compute the Aristotelian closure of a set of premises.*

There is another way to express the statement of Lemma 2.2: $\longrightarrow$ is a NOETHERIAN relation. This fact allows us to apply the famous NEWMAN - Lemma, well known for its central role in the framework of λ– calculus. The main property guaranteeing that a NOETHERIAN reduction relation $\longrightarrow$ allows us to compute *unique* normal forms, is the property of *local confluence* which we will define now.

First, let $\xrightarrow{*}$ denote the reflexive and symmetric closure of $\longrightarrow$. For two elements $\alpha, \beta \in \Sigma_f$ we will write $\alpha \downarrow \beta$ *iff* there exists $\gamma \in \Sigma$ such that $\alpha \xrightarrow{*} \gamma$ and $\beta \xrightarrow{*} \gamma$. Thus $\alpha \downarrow \beta$ *iff* α and β can be reduced – by individual sequences of applications of the rules of the system – to the same element γ.

[21] From now on, we will delete the index r on the relation sign $\longrightarrow$, as we now consider only the relation defined in Definition 2.4.

Definition 2.6. A reduction relation $\longrightarrow$ is called *local confluent* if, for each pair of elements β, γ which are related in such a way that there exists an α with $\beta \longleftarrow \alpha \longrightarrow \gamma$, it holds that $\beta \downarrow \gamma$.

Let us translate this property into the language of Aristotelian logic. Here $\beta \longleftarrow \alpha \longrightarrow \gamma$ signifies that a certain set of propositions, α, can be transformed into different sets β and γ by applying different syllogisms or immediate rules *or* by applying the same rule to different elements of α. (By the way, we know that β and γ each contain exactly one proposition more than α). Local confluence means that, under this condition on β and γ, these sets of propositions may be reduced to a common successor: $\beta \xrightarrow{*} \delta$, $\gamma \xrightarrow{*} \delta$. This sounds complicated, however, will easily be seen to hold true for the rule system of syllogisms and immediate rules.

Lemma 2.3. On Σ_f, the reduction relation defined by the syllogistic rule system is local confluent.

Proof. Let r_i, r_k be two of the 30 rules defining the Aristotelian syllogistic system, and suppose that, for a certain set α of propositions, $\alpha \xrightarrow{r_i} \beta$ and $\alpha \xrightarrow{r_k} \gamma$. Let us denote by δ the set of propositions which results by applying r_i to β: $\beta \xrightarrow{r_k} \delta$. Then it is easy to realise that $\gamma \xrightarrow{r_i} \delta$ (*commutativity* of syllogistic rule - application). Thus, local confluence has been shown to hold in this case. □

Lemma 2.4. (Newman - Lemma, c.f. [8]). Let $\longrightarrow$ be a Noetherian reduction on a set Σ. Then, the following conditions are equivalent:
(i) $\longrightarrow$ is local confluent
(ii) Each element has a unique normal form.

This implies that, for our formal setting of Aristotelian logic, the situation is as convenient as possible: Due to the conclusion of the Newman – Lemma, there is a unique normal form to each $\alpha \in \Sigma_f$, and it will be attained by applying the rules of the system in whichever order ! – We conclude by putting together all the parts into one theorem concerning the Aristotelian system of syllogistic logic.

Theorem 2.1. *Let Σ_f be defined as the set of all finite sets of propositions. Let r be the union of all syllogistic and immediate rules of Aristotelian logic (c.f. Definition 2.4). Then, for each premise set $\alpha = \{p_1, p_2, ..., p_n\}$, the following algorithm will always stop, and, independently of the order in which the rules are being applied, compute the unique normal form of α, which is the closure $\Phi(\alpha)$ of α:*

(1) k=0: Let $\alpha_0 = \alpha$.
(2) Let r_i be a rule in r which is applicable to α_k; let α_{k+1} denote *any* element of Σ_f such that $\alpha_k \xrightarrow{r_i} \alpha_{k+1}$.
(3) If, for some step k, *no* rule is anymore applicable to α_k, define $\Phi(\alpha) = \alpha_k$, and stop.

$\Phi : \Sigma_f \longrightarrow \Sigma_f$ *is the closure – (or consequence –) operator of Aristotelian logic. It assigns, to each set α of propositions, its 'Aristotelian closure' $\Phi(\alpha)$.*

Remark 2.2. All the properties which have been proven for the 'Aristotelian relation' r on Σ_f in this Section 2 *hold as well for any other subset of syllogistic and/or immediate rules!* This is because the theory presented in this section does not depend on any special properties of the rules $r_1, ..., r_{30}$. If we do not explicitly specify the relations entering into the definition of r or Φ, it is understood that the Aristotelian relation r of Definition 2.4 is meant.

3. The second syntactical level: Derivation of syllogisms

3.1. **Direct derivation.** In Section 1 we described how to arrive at the 256 syllogistic rules by means of simple combinatorical variations. As we have already mentioned, the Aristotelian method of differentiating between valid syllogistic rules (syllogisms) and invalid ones goes as follows: Aristotle assumed a small number of the rules as being evident (four as above, or, later, only two of them), and he showed how to derive the other syllogisms from these fundamental ones. This is his famous method of *Prior Analytics* which leads to a proof that exactly 24 out of the 256 syllogistic rules are syllogisms[22]. Thus *derivation* is a deductive method having syllogistic rules as

[22]In section 2 we considered these 24 syllogism (plus six immediate rules) as a *definition* of Aristotelian logic, and we did not ask for the reason why just these rules are the ones that are valid and why the other 232 are considered as being invalid – this will be the main subject from now on.

objects - not sets of propositions as in the basic level. ARISTOTLE'S arguments refer to both levels, and these levels have not always been correctly identified by his successors and interpreters.

We will now give a precise formulation of this deductive method, trying to stay close to what ARISTOTLE wrote.

ARISTOTLE considered a certain set of syllogisms as being basically given (perfect syllogisms), and he showed how to deduce the other valid ones from these basic syllogisms. His first approach was to consider the rules given in Table 5, containing four syllogisms of the first figure, as basically given. The aim now is to make a decision whether one of the other syllogistic rules, i.e. one of the 256 possibilities (minus, of course, the ones already contained in the system) has to be considered as (valid) syllogism or not. Let us assume that this syllogistic rule which is under scrutiny, has the label (n, U, V, W) $(n = 1, 2, 3, 4;\ U, V, W \in \{A, E, I, O\})$. Let (n, U, V, W) have the two premises p, q and the conclusion s.

Definition 3.1. (Method D of *Direct Derivation*)

(1) Define the set α of propositions as follows: $\alpha = \{p, q\}$. Thus, α contains exactly the premises of the rule to be proven[23].
(2) Apply all rules of the system to α until no rule is any longer applicable. Let the resulting set be denoted by α'.
(3) Check, whether the conclusion of the rule, s, is contained in α': If this is the case, then the rule is accepted as a (valid) syllogism.

Aristotle showed how to derive 8 syllogisms by the method D of direct derivation, utilizing his first system of Table 5 ([1], Table 2, p. 201). These are the additional syllogisms listed in Section 1.5 except for *Baroco (2,O,A,O)* and *Bocardo (3,A,O,O): Cesare, Camestres, Ferison, Festino, Darapti, Disamis, Datisi and Felapton.*

Since method D can be easily tranformed into a computer program, it is not difficult to check which syllogistic rules may be 'directly derived' from the four basic ones of Table 5: The computation not only produces the valid syllogisms but, checking all 256 syllogistic rules, also renders the information which of these rules are *not* valid[24]. Our first theorem which we have obtained by such a program[25], repeats a well known fact about the set of syllogisms.

Theorem 3.1. *Aristotle's rule system {E-con, A-pcon, I-con, Barbara, Celarent, Darii, Ferio; Direct derivation D} renders valid exactly 20 syllogistic rules out of the set of 256 syllogistic rules. This is the set of 24 classical syllogisms from Table 4 excluding Baroco, Bocardo, Camestrop and Camenop. This set of 20 syllogisms contains 12 Aristotelian syllogisms of Prior Analytics - i.e., the 14 syllogisms discussed there, minus Baroco and Bocardo.*

Camestrop and *Camenop* require the rule of E-subalternation (E-sub) in order to be shown valid by direct derivation. As A-pcon can be replaced by A-sub and I-con, we get the following

Corollary 3.1. *The rule system {E-con, I-con, A-sub, E-sub; Barbara, Celarent, Darii, Ferio; Direct derivation D} renders valid exactly 22 syllogistic rules out of the set of, in total, 256 syllogistic rules. This is the set of 24 classical Syllogisms from Table 4 excluding Baroco, Bocardo.*

We will call the syllogistic system {E-con, I-con, A-sub, E-sub ; Barbara, Celarent, Darii, Ferio, Baroco, Bocardo; Direct derivation D} the 'FSDB' (First Smallest Direct Basis) of the Aristotelian System, containing 4 immediate and 6 syllogistic rules. Up to now, we have of course did not proved that there is no direct basis with a smaller number of syllogisms; this will be the subject of Section 4. Furthermore, we will prove that there are exactly 64 'smallest' bases, out of which the FSDB is the first one in a certain natural lexicografic order.

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[23]To be quite precise, p and q are arbitrary *instances* of the premisses of the syllogistic rule to be proven: If, e.g., the rule is $Axy, Eyz \rightarrow Exz$, then we may take $p = At_1t_2$, $q = Et_2t_3$, $s = Et_1t_3$. Any other triple of different indices would also do.

[24]Thus we are given a purely syntactical 'method of rejection'.

[25]The result may of course be verified by hand calculation. For those who are used to relying on computer programs in formal reasoning, I have published an online version of my program at *www.aristotelianlogic.glashoff.net.* In addition, the results have been independently checked by a program written in *Mathematica* [14].

3.2. **Indirect derivation.** We have already mentioned that, in addition to direct derivation, Aristotle uses two different instruments for deriving syllogisms. One of these methods, which has a pivotal role in Aristotle's logic, is indirect derivation[26]. In order to precisely define this method, we have to specify the classical concepts of contradiction and contraries, which are part of the well known *Square of Oppositions*[27].

If p is a proposition, then the contradictory resp. contrary of p are are defined as in Table 6. The idea of indirect derivation for proving that a certain syllogistic rule (n, U, V, W) is a syllogism

Table 6. Contradictories and contraries

proposition p	**contradictory C(p)**	**contrary K(p)**
Axy	Oxy	Exy
Exy	Ixy	Axy
Ixy	Exy	-
Oxy	Axy	-

(valid syllogistic rule) works as follows.

Definition 3.2. (Method C of *Indirect derivation*)
(1) Let p, q, s denote the two premises and the conclusion of the syllogistic rule with label(n, U, V, W). Define $\alpha = \{p, q, C(s)\}$, where $C(s)$ is the contradictory of s according to Table 6.
(2) Apply the rules of the system (let us say, the Aristotelian rule system of Table 5) to α as long as possible, finally arriving at a set α' (which is the Aristotelian closure of α).
(3) Check, whether α' contains two contradictary propositions according to Table 6, i.e., a proposition of type Axy together with its contradictory proposition Oxy or a proposition of type Exy together with its contradictory Ixy.
(4) If the check has a positive result, the syllogistic rule (n, U, V, W) is valid.

There is a variant of this procedure which does not utilise contradictory propositions $C(s)$ but contraries $K(s)$ (Table 6):

Definition 3.3. (Method K of *Indirect derivation*)
- Steps (1), (2), and (4) work like Method C, only Step (3) has to be replaced by
- (3′): Check, whether α contains two *contrary* propositions according to Table 6, i.e., a proposition of type Axy together with its contrary proposition Exy.

The idea behind this procedure, invented by Aristotle, is to assume the contradictory of the conclusion together with the premises of the syllogistic rule to be proven, and to check whether this leads to a contradictory or contrary consequence. If this is the case, the syllogistic rule is valid.

Lemma 3.1. (Relation between *direct* derivation, method D and *indirect* derivations C and K)
- a) If it is possible to prove that a syllogism is valid (by using a certain set of immediate and syllogistic rules) by employing the method of direct derivation D, then it can also be shown to be valid by the method of indirect derivation C.
- b) If it is possible to prove that a syllogism is valid by employing the method of indirect derivation K, using a certain set of immediate and syllogistic rules which include the rules $E - sub$, or $A - sub$, or both $E - con$ and $A - pcon$, then it can also be shown to be valid by the method of indirect derivation C.

Proof. a) Let p, q, and s be the assumptions and the conclusion of the syllogism under scrutiny. Assume further, that it is possible to deduce s from p and q by direct derivation D (using a fixed set of rules). Then the closure of $\{p,\ q,\ C(s)\}$ (using the same set of rules) contains s as well as $C(s)$; thus the method C of indirect derivation also renders the result that the syllogism is valid.

[26]The other method is *ecthesis* which can be shown to be an alternative to indirect derivation ([16]).
[27]The Square of Oppositions appears already in the works of medieval authors like *William Sherwood* ([9]).

TABLE 7. Derivation method for syllogisms in Aristotle's second system

No.	Name[28]	Derivation	by means of	
1	Barbara*	—	Barbara	D
2	Celarent*	—	Celarent	D
3	Darii*	E-conv	Celarent	C
4	Ferio*	E-conv	Celarent	C
6	Celaront	—	Celarent	C
7	Cesare*	E-conv	Celarent	D
8	Camestres*	E-conv	Celarent	D
9	Festino*	-	Celarent	C
10	Baroco*	-	Barbara	C
11	Cesaro	-	Celarent	K
12	Camestrop	E-conv	Celarent	K
13	Darapti*	E-conv	Celarent	K
14	Disamis*	-	Celarent	C
15	Datisi*	E-conv	Celarent	C
16	Felapton*	E-conv	Celarent	K
17	Bocardo*	-	Barbara	C
18	Ferison*	E-conv	Celarent	C
19	Bamalip	E-conv	Celarent	K
20	Camenes	E-conv	Celarent	C
21	Dimatis	E-conv	Celarent	C
22	Fesapo	E-conv	Celarent	K
23	Fresison	E-conv	Celarent	C
24	Camenop	E-conv	Celarent	K

b) Let p, q, and s be as before and assume that the closure – with respect to the given set of rules – of $\{p, q, C(s)\}$ contains two contrary propositions t and *K(t).* This means that, for some terms x and y, the closure contains the propositions *Exy* and *Axy.* If the rule system includes the rule *E-sub*, then the proposition *Oxy* is contained in the closure, too, and, because *Axy* and *Oxy* are contradictory, the method of indirect derivation C applies. If the rule system includes the rule *A-sub*, the proposition *Ixy* is contained in the closure, and, because *Exy* and *Ixy* are contradictory, the method of indirect derivation C applies. If the rule system includes the rules *E-con* and *A-pcon*, the propositions *Eyx* and *Iyx* are contained in the closure, and, because *Eyx* and *Iyx* are contradictory, the method of indirect C-derivation applies. □

ARISTOTLE used indirect derivation in order to reduce the number of fundamental syllogisms: He showed that *Barbara* and *Celarent*, together with E-conversion, suffice for proving all other 12 syllogism which he considered to be valid. BOGER's Table 4 in [1] shows that ARISTOTLE'S proofs may be subsumed into the following theorem.

Theorem 3.2. ARISTOTLE'S *second rule system, {E-con; Barbara, Celarent; Direct method D, Indirect methods C and K}, renders valid the 14 Aristotelian Syllogisms.*

Our next step is to try out this same Aristotelian rule system for all 256 syllogistic rules; as this is just a small task for a 'brute force' method by means of a computer program, we only present the results of this computation.

Theorem 3.3. ARISTOTLE'S *second rule system, {E-con; Barbara, Celarent; Direct method D, Indirect methods C and K}, renders valid exactly the 24 classical Syllogism of Table 7; all other 232 syllogistic rules are rejected by this system.*

There is a modern system, given by SMILEY and, independently, by CORCORAN in 1972/73 ([15, 4]) which is different from the original Aristotelian one in not utilising indirect K - derivation, but using, in addition to *E-conv,* the immediate rule of *A-pcon*, $Axy \to Iyx$. According to Proposition 3.1, all syllogisms having a *'K '* in the last column of Table 7 can also be reduced by {*E-conv, A-pconv; Barbara, Celarent; Indirect C*}. The SMILEY/CORCORAN system also renders *invalid* all syllogistic rules *not* appearing in Table 7.

TABLE 8. Corcoran -Smiley system

1.	E-conversion	E-con
2.	(partial) A-conversion	A-pcon
2	Syllog. rule (1,A,A,A)	Barbara
3.	Syllog. rule (1,A,E,A)	Celarent

Therefore, the following theorem holds true:

Theorem 3.4. *The* SMILEY/CORCORAN - *system {E-conv, A-pconv; Barbara, Celarent; D, C} renders valid the 24 classical syllogisms and invalid all other 232 syllogistic rules.*

Thus, the second Aristotelian system, *{E-conv; Barbara, Celarent; D, C, K}* is equivalent to the CORCORAN-SMILEY system *{E-conv, A-pconv; Barbara, Celarent; D, C}* in that within both systems it is possible to derive exactly the same 24 syllogistic rules and prove exactly the same 232 remaining syllogistic rules to be invalid. Both systems are equivalent to MARTIN's system, *{E-conv, A-sub; Barbara, Celarent; D, C}* of [11] (by Lemma 3.1) and to the system *{E-conv, E-sub; Barbara, Celarent; D, C}* (again by Lemma 3.1)[29].

4. SYLLOGISTIC BASES

This Section is devoted to a more systematic look at the system of 24 syllogisms. In this section, we take these 24 rules as our objects of study, and we will analyse the structural properties of this set, which we will name *AR24*. There is a certain natural relation given on this set, namely, the derivation - relation: A syllogism Y is related to another syllogism X by derivation, if it is possible to show Y to hold by proving it by means of X together with certain immediate rules yet to be specified. In addition, the *proof method* has to be stated explicitly (D, C, and/or K).

Let us, for example, take E-conversion, I-conversion together with the direct derivation method D as tools. We define the relation $\overset{CS-D}{\Longrightarrow}$ on *AR24* as follows:

- $X \overset{CS-D}{\Longrightarrow} Y$ if and only if Y can be proven to be a syllogism (valid syllogistic rule) by derivation, using the following rule -system :*{E-con, I-con, A-sub, E-sub; syllogism X; Direct method D}.*

Another important relation is given by

- $X \overset{CS-C}{\Longrightarrow} Y$ if and only if Y can be proven to be a (valid syllogistic rule) by using the following rule-system: *{E-con, I-con, A-sub, E-sub;Syllogism X; Indirect method C}.*

Relations like $\overset{CS-D}{\Longrightarrow}$ and $\overset{CS-C}{\Longrightarrow}$ provide the set *AR24* of syllogism with quite natural *preorder structures* based on the derivation procedure invented by ARISTOTLE. This allows us to express important results of the theory of syllogism in terms of simple statements on preordered or partially ordered sets.

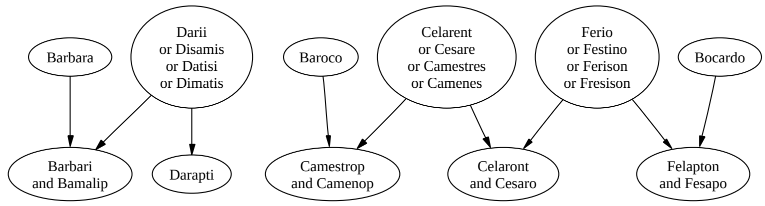
4.1. The set of 64 canonical 'direct bases'.

Let us begin with the relation $\overset{CS-D}{\Longrightarrow}$ because of its connection to Corollary 3.1 of Section 3.

Theorem 4.1. *AR24, equipped with the relation* $\overset{CS-D}{\Longrightarrow}$*, has the following structure, displayed in Fig. 4.1. In this Figure, the circles in the upper row enclose those syllogisms which are equivalent with respect to the relation* $\overset{CS-D}{\Longrightarrow}$*; i.e., for each two of the syllogisms X and Y within such a circle, we have* $X \overset{CS-D}{\Longrightarrow} Y$ *as well as* $Y \overset{CS-D}{\Longrightarrow} X$*. The circles in the lower row do not enclose equivalent syllogisms; they have been introduced in order to avoid an abundance of arrows.*

[29] All systems *{E-con; Barbara, Celarent; Direct method D, Indirect methods C and K}* (ARISTOTLE), *{E-conv, A-pconv; Barbara, Celarent; D, C}* (SMILEY/CORCORAN), *{E-conv, A-sub; Barbara, Celarent; D, C}* (MARTIN) and *{E-conv, E-sub; Barbara, Celarent; D, C}* can do without direct derivation D because of Part a) of Lemma 3.1.

FIGURE 4.1. Direct Syllogistic bases



Thus, the diagram must be read as follows: Barbari and Bamalip both 'follow' Barbara; i.e., Barbara $\stackrel{CS-D}{\Longrightarrow}$ *Barbari, Barbara* $\stackrel{CS-D}{\Longrightarrow}$ *Bamalip. The 'four D's', Darii, Disamis, Datisi, Dimatis are all equivalent with respect to* $\stackrel{CS-D}{\Longrightarrow}$*, and each of these syllogism implies Darapti. Baroco implies Camestrop and Camenop, which follow also from each of the equivalent syllogisms Celarent, Cesare, Camestres, and Camenenes; etc., etc.*

If, out of each of the circles of the upper row, one single syllogism is selected, *all* other syllogisms follow. This leads to the following

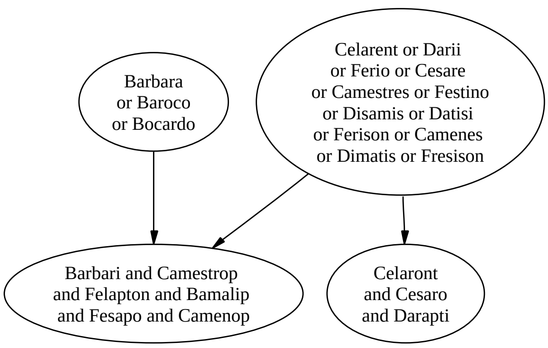
Corollary 4.1. *With respect to Direct derivation, together with the conversion and subalternation rules, the syllogistic system has a minimal base consisting of exactly six syllogisms. There are exactly 64 such bases, and the structure of the bases is as follows: Each base contains*
- *Barbara and Baroco and Bocardo*
- *one of Darii, Disamis, Datisi, Dimatis*
- *one of Celarent, Cesare, Camestres, Camenes*
- *one of Ferio, Festino, Ferison, Fresison.*

Let us relate this Corollary to Corollary 3.1: The rule system *{E-con, I-con, A-sub, E-sub ; Barbara, Celarent, Darii, Ferio, Baroco, Bocardo; Direct derivation D}* is one of these bases; it is the *first* base in the sense of a lexicographical order on the set of subsets of syllogisms or, to put it in another way, it contains the maximal number of syllogisms of the first figure. This is the reason why, in section 1, we called this set the 'First Smallest Direct Basis'.

4.2. **The set of 36 indirect bases.** Now we consider indirect derivation, together with simple conversion and subalternation; i.e. we let *AR24* be equipped by the relation which is defined by *{E-con, I-con, A-sub, E-sub; Indirect Method C}*. A computational exploration of the set *AR24* of all syllogisms results in the following theorem.

Theorem 4.2. *AR24, equipped with the relation* $\stackrel{CS-C}{\Longrightarrow}$*, has the following structure:*

FIGURE 4.2. "Indirect" syllogistic bases



Again, the upper circles contain syllogisms which are equivalent WRT the preorder $\stackrel{CS-C}{\Longrightarrow}$ *, while the syllogisms below are not equivalent but are grouped only in order to avoid an abundance of arrows. Thus, for example, the leftmost arrow has to be understood as follows: Each of the syllogisms Barbara, etc., contained within the leftmost upper circle, implies – wrt. to the relation*

$\overset{CS-C}{\Longrightarrow}$ – *all the six syllogisms within the leftmost lower circle. In other words: Barbari, Camestrop, etc. may be derived by either Barbara or Baroco or Bocardo by employing the rules of conversion and subalternation, using the method of indirect C–derivation.*

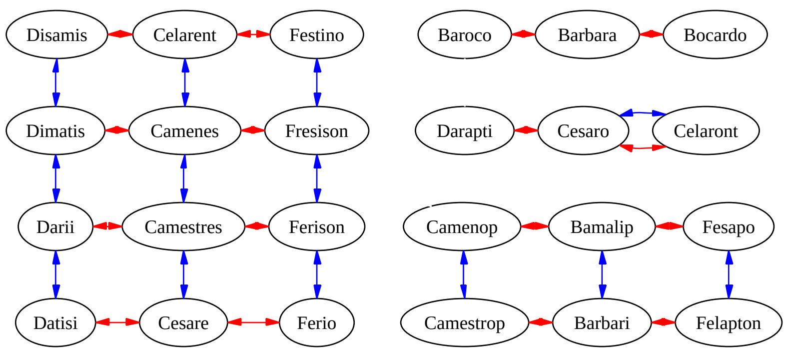
Corollary 4.2. *With respect to indirect C-derivation together with the conversion and subalternation rules, the syllogistic system has a minimal base consisting of exactly two syllogisms. There are exactly* $3 * 12 = 36$ *such bases, and the structure of the bases is as follows: Each base contains*
- one of the three syllogisms Barbara, Baroco, Bocardo
- one of the 12 syllogisms Celarent, Darii, Ferio, Cesare, Camestres, Festino, Disamis, Datisi, Ferison, Camenes, Dimatis, Frerison.

Within this set of bases, there are three bases consisting of syllogisms of the first figure only; the most prominent of these is the 'First minimal indirect basis' *{E-con,I-con,A-sub,E-sub; Barbara, Celarent; Indirect method C}*. Again, to put it another way: Each of the 24 syllogisms can be reduced to *Barbara* or *Celarent*, utilising indirect C--derivation together with the four rules of simple conversion and subalternation.[30]

Another possibility, taken by Smiley ([15]) in 1973, was to use *{E-con, A-pcon;D,C}*; an easy calculation shows that Theorem 4.2 also holds in that case. This observation connects Theorem 3.4 with the theory in this section, especially with the foregoing corollary.

The next graphic[31] integrates two different relations: The vertical arrows denote equivalence with respect to direct derivation, utilising only simple conversion, i.e. $\overset{Conv-D}{\Longrightarrow}$, while the horizontal arrows denote equivalence with respect to pure indirect derivation, i.e., $\overset{C}{\Longrightarrow}$. There is one exception: *Cesaro* and *Celaront* can be derived from each other by both methods.

Figure 4.3. Direct-Indirect Bases with Conversion



Again, this graphic may be interpreted in terms of bases of the syllogistic system: It says, that there are exactly $12 * 6 * 3 * 3 = 648$ different bases of which each consists of four syllogisms, if one allows the method of direct D–derivation *and* indirect C–derivation together with simple conversion *{Econ, Icon}*. It is interesting that the Aristotelian set of four perfect syllogisms of the first kind, *{Barbara, Celarent, Darii, Ferio}* is *not* one of these bases; the reason is that Aristotle additionally used partial A-conversion (*A-pcon*).

4.3. **Systems with identity.** In his *Prior Analytics*, Aristotle did not explicitely state axioms of type Axx or Ixx .[32] In later times, these identities or have frequently been added to the Aristotelian

[30]This may be considered as the formal specification of the "*dictum de omni et nullo*" ([7], p. 296) which says: "What is predicated (stated or denied) about any whole is predicated (stated or denied) about any part of that whole."

[31]This graphic is similar to that of Thom ([20] p.98). It is presented here in order to indicate the wealth of structural information which we are able to obtain with our method. I hope that - by this picture - the structural aspects of the syllogistic system are easier to understand than by the one given in Thom's book. It must be said, however, that Thom added two other relations to his graphics which makes it nearly impossible to catch the essence of the overall picture.

[32]As within the framework of Aristotelian logic, Axy may be interpreted as "y is a genus for the species x", an expression like Axx does not make sense.

system; for example by LEIBNIZ (c.f. Parkinson, [12], p. 105 – 111). We will denote these axioms as

(1) A-Identity: For each term x, add Axx to the set of given propositions.
(2) I-Identity: For each term x, add Ixx to the set oft given propositions.

LEIBNIZ found out that A-Identity, together with certain syllogisms, could replace the conversion and subalternation rules. For example, E-conversion can be substituted by A-Identity plus *Cesare*, if one allows premises with identical terms in syllogisms.[33] Applying *Cesare* in this manner, we obtain the rule

$$Axx, Ezx \longrightarrow Exz$$

Thus it is of some interest to consider rule systems of type *A-Id. + syllogisms + direct/indirect method* as an alternative to the systems considered in the previous sections. We obtained the following theorem by a brute-force method, trying out all different combinations of sets of syllogisms with up to 7 elements each, together with A-Identity and method D of derivation:

Theorem 4.3. *(Indirect syllogistic bases)*[34]

(1) *There is no set consisting of less than 7 syllogisms which, together with A-Identity and the method of direct derivation, allows derivation of all 24 syllogisms.*
(2) *There are exactly 48 sets of syllogisms containing 7 syllogisms with the following property: By means of each set, together with A-Identity and the method of direct derivation,* exactly *the 24 syllogisms of Table 4 may be derived.*
(3) *The bases have the following structure: They contain*
   - *Barbara and Baroco and Bocardo*
   - *Ferio or Festino or Ferison or Fresison*
   - *Babari or Darapti or Bamalip*
   - *Cesare or Camenes*
   - *Datisi or Dimatis*

The first basis[35] of this type, containing the highest possible number of syllogisms of the first figure and no syllogism of the fourth figure, is *{Barbara, Baroco, Bocardo, Ferio, Barbari, Disamis, Cesare}*. The "last basis" (with maximal index-sum) of these 48 sets is *{Barbara, Baroco, Bocardo, Bamalip, Camenes, Dimatis, Fresison}*. This special set has been found by BRILLOWSKI ([3], p. 129).

5. CONCLUSION AND PERSPECTIVES

In this paper we have presented a simple algorithm–oriented theory of Aristotelian syllogistic logic. We managed to formalise this logic on a purely syntactic level by means of the theory of reduction relations on the set of subsets of Aristotelian propositions. This in turn led to a precise formalisation of the process of *derivation of syllogisms* which is a predominant part of the traditional syllogistic theory. We embedded the theory of derivation into our theory of syllogistic bases which employs different preorder structures on the set *AR24* of all syllogisms. Thus we were able to obtain and countercheck a wealth of classical structural results, ranging from the times of ARISTOTLE to today's results of CORCORAN, SMILEY, and THOM.

There are some subsequent fields for further research in which the method developed in this paper can be applied or generalised:

(1) *Formalisation of ecthesis.* Ecthesis is a proof method which, in his *Prior Analytics*, ARISTOTLE used in addition to indirect C-derivation. It is possible to consider ecthesis as a syntactical tool (c.f. Robin SMITH, [16]), and it is as easy to formalizse as the other deduction methods within our rule–based system.
(2) *Negated terms.* There is a classical variant of ARISTOTLE'S logic which uses the notion of negated terms (non–man, non–animal etc.). The negation operator can easily be introduced

[33] Definition 2.1 must be changed appropriately in order to allow propositions with identical terms.

[34] This theorem is related to a statement made by THOM on p.104. However, there is one difference concerning "Barbari or Darapti or Bamalip", as THOM would also allow Camenop or Fesapo or Felapton or Cesaro or Celaront instead of "Barbari or Darapti or Bamalip", which I could not reproduce. For example, I do not see how it might be possible to derive Darapti, if neither Barbari nor Bamalip belong to the basic set of syllogisms.

[35] in the sense that the sum of the indices of its members is minimal; cf. Table 4.

into our formalism, and the corresponding structural properties can be explored by a slight extension of the methods presented in Section 4 of this paper.

(3) *Characterisation of the set of syllogisms.* In our paper we have shown how to characterise the set *AR24* of valid syllogisms: It is the closure, within the set of 256 syllogistic rules, of certain 'small' sets of rules by means of direct or indirect derivation. However, we did not search for an optimal algebraic characterisation of *AR24* – this would require a more detailed study of the set of 256 rules, equipped with a suitable structure. The question is: *What kind of structure* would have to be imposed onto the set of 256 rules, in order to render *AR24* as the natural candidate for a valid rule set?

Let us close with some remarks on the semantics of ancient Greek logic. ARISTOTLE'S logic is certainly not an exclusively syntactically designed system. However, logicians and philosophers do not agree *which* intended interpretation ARISTOTLE and his classical followers had in mind. The crucial question is: What kind of entities do terms stand for? Do they stand for nonempty *sets* of individuals (modern *extensional* interpretation) or do they designate *concepts (intensional* interpretation*)*. Both interpretations can be taken as a basis for a semantical theory of Aristotelian logic. Without trying to enter into a discussion of this, still controversial, subject, let us just note that the *intensional* way of interpreting Aristotelian logic is very much in the spirit of many authors from ARISTOTLE himself up to LEIBNIZ, KANT and ŁUKASIEWICZ, to name only a few. It was ŁUKASIEWICZ (see [6]) who used LEIBNIZ' intensional arithmetical semantics as a means for proving a completeness theorem for Aristotle's logic. It will be a challenging task to do the same also for the formal system presented in our paper, and the decision *not* to settle this question here is only due to a self imposed restriction of remaining strictly on the syntax level.

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REFERENCES

[1] George Boger. Completion, Reduction and Analysis: Three Proof-theoretic Processes in Aristotle's Prior Analytics. *History and Philosophy of Logic*, 19:187–226, 1998.

[2] George Boger. Aristotle's Underlying Logic. In Dov M. Gabbay and John Woods, editors, *Handbook of the History of Logic*, volume 1, pages 101 – 246. Elsevier North Holland, Amsterdam, 2004.

[3] Claus Brillowski. *Grundlagen offener Sprachen.* Europäische Hochschulschriften. Peter Lang, Frankfurt am Main, 1993.

[4] J. Corcoran. Completeness of an Ancient Logic. *Journal of Symbolic Logic*, 37(4):696–702, 1973.

[5] Kurt Ebbinghaus. *Ein formales Modell der Syllogistik des Aristoteles.* Number 9 in Hypomnemata. Vandenhoeck & Rupprecht, Göttingen, 1964.

[6] Klaus Glashoff. On Leibniz' characteristic numbers. *Studia Leibnitiana*, 34:161, 2002.

[7] H. W. B. Joseph. *An Introduction to Logic.* The Paper Tiger, Cresskill, NJ, 2 edition, 2000. First published by Oxford University Press, 1916.

[8] J. W. Klop. Term Rewriting Systems. In D. M. Gabbay and S. Abramsky, editors, *Handbook of Logic in Computer Science*, volume 2, pages 2 – 117. Clarendon Press, 1992.

[9] N. Kretzmann. *William Sherwood's Introduction to Logic. Translation with an Introduction and Notes.* Minneapolis, 1966.

[10] Jan Łukasiewicz. *Aristotle's Syllogistic.* Clarendon Press, Oxford, 2. edition, 1957.

[11] J. N. Martin. Aristotle's Natural Deduction Reconsidered. *History and Philosophy of Logic*, 18:1–15, 1997.

[12] G. H. R. Parkinson. *Leibniz: Logical Papers. A Selection Translated and Edited with an Introduction.* Oxford University Press, Oxford, 1966.

[13] G. Patzig. *Aristotle's Theory of the Syllogism.* Reidel, Dordrecht, 1968.

[14] Wolfram Research. *Mathematica.* Champaign, IL., 2002. Version 4.2.

[15] T. Smiley. What is a Syllogism? *Journal of Philosophical Logic*, 2:136–154, 1973.

[16] R. Smith. An Ecthetic Syllogistic. *Notre Dame Journal of Formal Logic*, 24:224–232, 1983.

[17] Robin Smith. Aristotle's Logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* http://plato.stanford.edu/archives/fall2003/entries/aristotle-logic/, Fall 2003.

[18] John F. Sowa. *Knowledge Representation: Logical, Philosophical, and Computational Foundations.* Brooks Cole Publishing Co., Pacific Grove, CA, 2000.

[19] P. F. Strawson. *Introduction to Logical Theory.* University Paperbacks. Methuen, 1952.

[20] Paul Thom. *The Syllogism.* Philosophia Verlag, München, 1981.

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